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Phase space structure of Chern-Simons theory with a non-standard puncture

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Abstract

We explicitly determine the symplectic structure on the phase space of Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$ on a three-manifold of topology $\mathbb{R} \times S^\infty_{g,n}$, where $S^\infty_{g,n}$ is a surface of genus g with $n+1$ punctures. At each puncture additional variables are introduced and coupled minimally to the Chern-Simons gauge field. The first n punctures are treated in the usual way and the additional variables lie on coadjoint orbits of $G \ltimes \mathfrak{g}^*$. The $(n+1)$st puncture plays a distinguished role and the associated variables lie in the cotangent bundle of $G \ltimes \mathfrak{g}^*$. This allows us to impose a curvature singularity for the Chern-Simons gauge field at the distinguished puncture with an arbitrary Lie algebra valued coefficient. The treatment of the distinguished puncture is motivated by the desire to construct a simple model for an open universe in the Chern-Simons formulation of $(2+1)$-dimensional gravity.

1 Introduction

Chern-Simons field theory has attracted the attention of both mathematicians and physicists. Its relevance in mathematics is largely due to its applications in fields such as knot theory, see [1] or the book [2], and the theory of moduli spaces of flat connections, see [3] for a summary. These research areas in turn provide useful concepts and methods for the study of Chern-Simons theory. From a physicist's point of view, Chern-Simons theory is interesting because it captures important aspects of real physical systems and is at the same

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time mathematically tractable. While not describing any real physical system, it plays an important role in, for example, the modelling of certain condensed matter systems. More importantly, it shares structural features with fundamental physical theories. Its diffeomorphism invariance, for instance, makes it a useful toy model for the investigation of coordinate independent approaches to quantisation. In particular, it is relevant to the study of Einstein's theory of gravity, which in (2+1) dimensions can be formulated as a Chern-Simons theory [4, 5].

The basic reason for the relative mathematical simplicity of Chern-Simons theory is that, with appropriate boundary conditions and after dividing out gauge degrees of freedom, it has a finite-dimensional phase space. Thus, in suitable circumstances Chern-Simons theory allows one to reduce the field-theoretic description of a physical system to a mathematically well-defined and finite-dimensional model. This reduction from an infinite to a finite number of degrees of freedom makes the classical theory tractable and considerably simplifies the quantisation.

In this paper we study Chern-Simons theory in its Hamiltonian formulation on a manifold of topology $\mathbb{R} \times S^\infty_{g,n}$, where $S^\infty_{g,n}$ is a surface of genus g with $n+1$ punctures. The purpose of the paper is to introduce a new way of treating punctures in Chern-Simons theory and to determine the physical phase space and its symplectic structure when one distinguished puncture is treated in this way. The main result is an explicit determination of the symplectic structure on the finite-dimensional physical phase space.

The motivation for our treatment of the distinguished puncture comes from the Chern-Simons formulation of (2+1)-dimensional gravity. As we explain in detail in a separate paper [6], the distinguished puncture can be used to model "spatial infinity" in open universes. Applied to (2+1)-dimensional gravity, our model leads to a finite-dimensional description of the phase space which can serve as a starting point for both an investigation of the classical dynamics and for quantisation, using the methods developed in [7]. The relation to (2+1)-dimensional gravity is also the motivation for our choice of gauge group. We consider gauge groups of the form $G \ltimes \mathfrak{g}^*$, where G is an arbitrary finite-dimensional Lie group. They include as special cases the Euclidean group and the Poincaré group in three dimensions which arise in the Chern-Simons formulation of Euclidean and Lorentzian (2+1)-gravity with vanishing cosmological constant [5]. Mathematically, groups of the form $G \ltimes \mathfrak{g}^*$ are particularly simple examples of Poisson-Lie groups, which is important in our analysis. We stress, however, that our treatment of the distinguished puncture and many of our results concerning the phase space are not limited to gauge groups of this type.

Since much of the paper is quite technical, we give a brief sketch of our treatment of punctures on $S^\infty_{g,n}$. The usual approach, followed in [1] and summarised in [3], is to require the curvature of the gauge field to have a delta-function singularity on the line $\mathbb{R} \times \{x_{(i)}\}$, where $x_{(i)}$ is the coordinate of the puncture on $S^\infty_{g,n}$, and to restrict the Lie algebra valued coefficient of the delta-function to a fixed coadjoint orbit of the gauge group. In order to achieve this, additional variables need to be introduced which parametrise the coadjoint orbit. The

dynamics of these additional variables is governed by the Kirillov-Kostant-Souriau symplectic structure on the coadjoint orbit minimally coupled to the Chern-Simons gauge field. In this paper we treat the first n punctures on $S^\infty_{g,n}$ in this way. At the distinguished puncture, whose coordinate on $S^\infty_{g,n}$ is denoted x_∞, we also require the curvature to have a delta-function singularity, but this time we do not restrict the Lie algebra valued coefficient T. Instead we introduce an additional Lie group valued variable g at the distinguished puncture and interpret the pair (g, T) as an element of the cotangent bundle of the gauge group. The dynamics of the variables g and T is governed by the canonical symplectic structure on the cotangent bundle minimally coupled to the Chern-Simons gauge field.

Having defined the field-theoretical model we parametrise the physical phase space as a finite-dimensional quotient of a finite-dimensional extended phase space and compute the pull-back of the symplectic structure to the extended phase space. Our calculations make extensive use of the results and techniques of [8]. In that paper, Alekseev and Malkin consider Chern-Simons theory with a simple, complex gauge group (or its compact real form) on a three-manifold of the form $\mathbb{R} \times S_{g,n}$, where $S_{g,n}$ is a surface of genus g with n punctures which are treated in the usual way. Here we extend their analysis to include the distinguished puncture and to apply to gauge groups of the form $G \ltimes \mathfrak{g}^*$, where G is any real Lie group. In particular, we do not assume the existence of a single abelian Cartan subalgebra into which any Lie algebra element can be conjugated; such a Cartan subalgebra plays an important role in [8]. However, the semi-direct product structure of $G \ltimes \mathfrak{g}^*$ also leads to simplifications. It allows us to give a more detailed formula for pull-back of the symplectic form to the extended phase space and to show that this pull-back is exact.

The paper is organised as follows. After explaining our notation and conventions in Sect. 2, we define our field-theoretical model by giving the action and equations of motion in Sect. 3. The rather technical Sect. 4 explains the adaptation of Alekseev and Malkin's method to our situation and ends in a first formula for the pull-back of the symplectic form to the extended phase space, valid for any gauge group. In Sect. 5 we use the structure of the gauge group $G \ltimes \mathfrak{g}^*$ to derive the formula for the pull-back of the symplectic form as an exterior derivative of a one-form. In Sect. 6 we explain in detail how the physical phase space is obtained from the extended phase space by imposing various constraints and dividing by the associated gauge transformations. Sect. 7 contains a short discussion and conclusion, and the Appendices A and B, respectively, summarise the properties of groups $G \ltimes \mathfrak{g}^*$ as Poisson-Lie groups and key results from [8] needed in the main part of the paper.

2 Setting and Notation

Let M be a three-manifold of topology $\mathbb{R} \times S^\infty_{g,n}$, where $S^\infty_{g,n}$ is an oriented two-dimensional manifold of genus g with $n+1$ punctures. Of these, one will play a special role, and we refer to it as the distinguished puncture to differentiate it from the remaining n ordinary punctures. We introduce a global coordinate x^0 on $\mathbb{R}$, write $x = (x^1, x^2)$ for local coordinates on the surface $S^\infty_{g,n}$ and denote the differentiation with respect to x^0, x^1 and x^2 by ∂_0, ∂_1 and ∂_2.

The coordinates of the n ordinary punctures on $S^\infty_{g,n}$ are $x_{(1)},\ldots,x_{(n)}$, and the distinguished puncture has the coordinate x_∞. We refer to the one-dimensional submanifolds defined by $x = x_{(i)}$, $i = 1,\ldots,n$, and $x = x_\infty$ as, respectively, the world lines of the ordinary punctures and the distinguished puncture.

Chern-Simons theory is a field theory for an H-connection on M. For now, H can be an arbitrary finite-dimensional Lie group, but we will restrict ourselves to a particular class of Lie groups further below. Locally, the connection is given by a Lie algebra valued one-form A on M, called the gauge field. The product structure $M = \mathbb{R}\times S^\infty_{g,n}$ allows us to decompose the gauge field as

$$A = A_0 dx^0 + A_S, \tag{2.1}$$

where A_S is an x^0-dependent and Lie algebra valued one-form on $S^\infty_{g,n}$ and A_0 is a Lie algebra valued function on $\mathbb{R}\times S^\infty_{g,n}$. In the following we denote by d the exterior derivative on $\mathbb{R}\times S^\infty_{g,n}$. Occasionally, we need to differentiate with respect to the dependence on $S^\infty_{g,n}$ only and denote such derivatives by d_S. With this notation, the curvature of the connection A is the two-form

$$F = dA + A\wedge A, \tag{2.2}$$

and the splitting (2.1) leads to the decomposition

$$F = dx^0\wedge(\partial_0 A_S - d_S A_0 + [A_0, A_S]) + F_S, \tag{2.3}$$

where F_S is the curvature two-form on $S^\infty_{g,n}$:

$$F_S = d_S A_S + A_S\wedge A_S. \tag{2.4}$$

In the following, we consider gauge groups that are semi-direct products $H = G\ltimes\mathfrak{g}^*$ of a connected, finite-dimensional Lie group G and the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g} = \mathrm{Lie}\,G$, viewed as an abelian group. The group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint action Ad and, following the conventions of [9], we define $\mathrm{Ad}^*(g)$ as the algebraic dual of $\mathrm{Ad}(g)$, i. e.

$$\langle \mathrm{Ad}^*(g)\boldsymbol{j},\xi\rangle = \langle \boldsymbol{j},\mathrm{Ad}(g)\xi\rangle \qquad \forall \boldsymbol{j}\in\mathfrak{g}^*, \xi\in\mathfrak{g}, g\in G, \tag{2.5}$$

where $\langle,\rangle$ is the canonical pairing of elements of $\mathfrak{g}$ with elements of $\mathfrak{g}^*$. Note that with this definition the coadjoint action of $g\in G$ is given by $\mathrm{Ad}^*(g^{-1})$. Writing elements of $G\ltimes\mathfrak{g}^*$ as $(u,\boldsymbol{a})$ with $u\in G$, $\boldsymbol{a}\in\mathfrak{g}^*$, the group multiplication in $G\ltimes\mathfrak{g}^*$ is

$$(u_1,\boldsymbol{a}_1)\cdot(u_2,\boldsymbol{a}_2) = (u_1\cdot u_2, \boldsymbol{a}_1 + \mathrm{Ad}^*(u_1^{-1})\boldsymbol{a}_2). \tag{2.6}$$

In this paper, all Lie algebras will be considered over $\mathbb{R}$. The Lie algebra of $G\ltimes\mathfrak{g}^*$ is $\mathfrak{g}\oplus\mathfrak{g}^*$ as a vector space, and we denote it as $\mathfrak{g}\tilde{\oplus}\mathfrak{g}^*$ to remind the reader that it is not the direct sum of $\mathfrak{g}$ and $\mathfrak{g}^*$ as a Lie algebra. The Lie bracket can be characterised as follows. Let J_a, $a = 1,\ldots,\dim G$, be a basis of the Lie algebra $\mathfrak{g}$ and let P^a, $a = 1,\ldots,\dim G$, be the dual

basis of $\mathfrak{g}^*$ i.e. $\langle P^b, J_a\rangle = \delta^b_a$. Then J_a, P^b, $a, b = 1, \ldots, \dim G$, form a basis of $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ with commutators

$$[J_a, J_b] = f_{ab}{}^c\, J_c \qquad [J_a, P^b] = -f_{ac}{}^b\, P^c \qquad [P^a, P^b] = 0, \tag{2.7}$$

where $f_{ab}{}^c$ are the structure constants of $\mathfrak{g}$.

The pairing $\langle , \rangle$ between $\mathfrak{g}$ and $\mathfrak{g}^*$ can be extended to a symmetric, non-degenerate bilinear form on $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$, also denoted by $\langle , \rangle$, by setting

$$\langle \xi, \boldsymbol{j}\rangle = \langle \boldsymbol{j}, \xi\rangle, \quad \text{and} \quad \langle \boldsymbol{j}, \boldsymbol{k}\rangle = \langle \xi, \eta\rangle = 0, \quad \boldsymbol{j}, \boldsymbol{k} \in \mathfrak{g}^*,\ \xi, \eta \in \mathfrak{g}. \tag{2.8}$$

This pairing is $G \ltimes \mathfrak{g}^*$-invariant by virtue of (2.5), and with this pairing the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ is canonically isomorphic to its dual. We will use this isomorphism to identify $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ with its dual without writing it explicitly in the following; both will be denoted by $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$. In particular, we write both the adjoint and the coadjoint action of an element $h \in G \ltimes \mathfrak{g}^*$ on $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ simply by conjugation with h.

In our discussion of the behaviour of the Chern-Simons gauge field at the puncture we make use of coadjoint or, equivalently, adjoint orbits of the $G \ltimes \mathfrak{g}^*$. It is then convenient to parametrise these orbits in the form $\{hDh^{-1} | h \in G \ltimes \mathfrak{g}^*\}$, where D is a fixed element of the Lie algebra. When dealing with Lie algebras over $\mathbb{C}$ or compact forms of complex Lie algebras as in [8], it is possible to fix one Cartan subalgebra of the Lie algebra and choose D to lie in that Cartan subalgebra without loss of generality. However, for general Lie algebras over $\mathbb{R}$ the theory of Cartan subalgebras, defined as nilpotent subalgebras which are their own normaliser, is more complicated. There are two features which are important for us.

The first is related to the conjugacy of Cartan subalgebras. As explained in [10] or [11], Cartan subalgebras of real Lie algebras can not necessarily be conjugated into each other. Instead, there exists a family of Cartan subalgebras $\mathfrak{c}_\iota$, $\iota \in I$, where I is some finite index set, such that any Cartan subalgebra is conjugate to one of the $\mathfrak{c}_\iota$. Moreover, not all elements of the Lie algebra lie in a Cartan subalgebra. This is only the case for regular elements[3]. A regular element can therefore always be written as $T = hDh^{-1}$, where D is in one of a finite number of fixed Cartan subalgebras.

The second issue we need to be aware of is that a Cartan subalgebra of a real Lie algebra is not necessarily abelian. In order to make sure that all Cartan subalgebras of $G \ltimes \mathfrak{g}^*$ are abelian we need to assume that G is semi-simple. This can be seen as follows. Cartan subalgebras of semi-simple Lie algebras are abelian [12]. Furthermore, if we assume that $\mathfrak{g}$ is semi-simple it follows from Theorem 9.5 in chapter 1 of [12] applied to the Levi decomposition $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ that Cartan subalgebras of $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ are of the form $\mathfrak{c} \oplus \mathfrak{c}^*$, with $\mathfrak{c}$ being a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{c}^*$ a Cartan subalgebra of $\mathfrak{g}^*$. However, such subalgebras of $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ are automatically abelian.

[3] By definition, a regular element T of a Lie algebra is such that the multiplicity of the characteristic root of $\mathrm{ad}T$ is equal to the rank of the Lie algebra.

With these structural features in mind, we adopt the following conventions for Cartan subalgebras. In order to parametrise regular elements we pick a finite family of Cartan subalgebras $\mathfrak{c}_\iota, \iota \in I$ of $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ for and write $T_{\mathfrak{c}_\iota}$ for the subgroup obtained by exponentiating the Cartan subalgebra $\mathfrak{c}_\iota$. We assume the existence of at least one abelian Cartan subalgebra and denote it by $\mathfrak{c}$; the abelian subgroup obtained by exponentiating $\mathfrak{c}$ is denoted $T_{\mathfrak{c}}$.

The group $G \ltimes \mathfrak{g}^*$ has the structure of a Poisson-Lie group, which is described in detail in [7] and partly reviewed in Appendix A. Poisson-Lie groups have compatible Poisson and Lie group structures, and for every Poisson-Lie group there is a dual Poisson-Lie group where Lie and Poisson structure are, in a suitable sense, interchanged. As a group, the dual of $G \ltimes \mathfrak{g}^*$ is the direct product $G \times \mathfrak{g}^*$. The Poisson-Lie group structure gives rise to a diffeomorphism between $G \ltimes \mathfrak{g}^*$ and its dual $G \times \mathfrak{g}^*$ which is given by (A.12) in Appendix A . The practical use of this diffeomorphism for our calculations is a parametrisation of elements in $G \ltimes \mathfrak{g}^*$ in terms of elements of $G \times \mathfrak{g}^*$. Writing an element of $G \times \mathfrak{g}^*$ as $(u, -\boldsymbol{j})$ the corresponding element in $G \ltimes \mathfrak{g}^*$ is given by

$$(u, \boldsymbol{a}) = (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \qquad \text{with } u \in G,\ \boldsymbol{a}, \boldsymbol{j} \in \mathfrak{g}^*. \tag{2.9}$$

3 Action and equations of motion

In order to define Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$ on a surface with punctures we need to specify the behaviour of the gauge field near the punctures. The standard way of including punctures in Chern-Simons theory is to allow the gauge field to have singularities of a particular form at the punctures. The singularities are such that the curvature (2.4) of the spatial gauge field A_S develops delta-function singularities with Lie algebra valued coefficients restricted to fixed coadjoint orbits of $G \ltimes \mathfrak{g}^*$. However, motivated by the application to the Chern-Simons formulation of (2+1)-dimensional gravity, see [6], we now formulate an action functional which allows for the inclusion of an additional puncture where the restriction to fixed coadjoint orbits is not imposed.

We start by summarising the usual treatment of punctures in Chern-Simons theory [1], adapted to gauge groups of the form $G \ltimes \mathfrak{g}^*$. For Chern-Simons theory on a genus g surface $S_{g,n}$ with n punctures at coordinates $x_{(1)}, \ldots, x_{(n)}$, one requires that the spatial curvature (2.4) takes the form

$$\tfrac{k}{2\pi} F_S(x) = \sum_{i=1}^{n} T_i \delta^{(2)}(x - x_{(i)}) dx^1 \wedge dx^2 \tag{3.1}$$

with Lie algebra valued coefficients $T_i \in \mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ restricted to certain coadjoint orbits of $G \ltimes \mathfrak{g}^*$. This restriction is imposed to ensure that the phase space of the theory is symplectic [1]. We assume in the following that the elements T_i are regular. This assumption simplifies the notation and saves us having to distinguish cases in the discussion. Moreover, it is satisfied in the application to (2+1)-dimensional gravity which is our main motivation. As explained in the previous section, each regular element T_i can then be conjugated into one of a family

of Cartan subalgebras $\mathfrak{c}_\iota$, with ι ranging over a finite index set I. Thus we write each T_i in terms of $h_i \in G \ltimes \mathfrak{g}^*$ and a fixed elements $D_i \in \cup_{\iota \in I} \mathfrak{c}_\iota$ as

$$T_i = h_i D_i h_i^{-1}. \tag{3.2}$$

For a more detailed description and an explicit parametrisation of coadjoint orbits of $G \ltimes \mathfrak{g}^*$ we refer the reader to Appendix A.1.

The action for Chern-Simons theory coupled to punctures contains the Chern-Simons action for the gauge field A and kinetic terms for the orbit variables $h_i \in G \ltimes \mathfrak{g}^*$ which are derived from the Kirillov-Kostant-Souriau symplectic structure on the coadjoint orbits and coupled to the gauge field via minimal coupling. The product structure $\mathbb{R} \times S_{g,n}$ allows us to give the action in its Hamiltonian form, which makes use of the decomposition (2.1):

$$\begin{aligned} S[A_S, A_0, h_i] = &\int_{\mathbb{R}} dx^0 \int_{S_{g,n}^\infty} \tfrac{k}{4\pi} \langle \partial_0 A_S \wedge A_S \rangle - \int_{\mathbb{R}} dx^0 \sum_{i=1}^n \langle D_i \, , \, h_i^{-1} \partial_0 h_i \rangle \\ &+ \int_{\mathbb{R}} dx^0 \int_{S_{g,n}^\infty} \langle A_0 \, , \, \tfrac{k}{2\pi} F_S - \sum_{i=1}^n T_i \delta^{(2)}(x - x_{(i)}) dx^1 \wedge dx^2 \rangle. \end{aligned} \tag{3.3}$$

Note that A_0 plays the role of a Lagrange multiplier and that the fixed elements D_i enter as parameters.

We now include an additional distinguished puncture labelled by ∞, at which the curvature also develops a Lie algebra valued singularity, but this time without restriction on the Lie algebra element multiplying the delta-function. In order to impose the desired curvature singularity we introduce a further $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$-valued variable T (not restricted to lie on a particular coadjoint orbit) as well as a group valued variable $g \in G \ltimes \mathfrak{g}^*$ which will play the role of the conjugate variable to T. Similar to the case of the ordinary punctures, the component A_0 should again act as a Lagrange multiplier imposing the constraint on the spatial curvature (2.4), and there should be a dynamical term involving the variables T and g. However, in contrast to the ordinary punctures, where the kinetic terms are obtained from the canonical symplectic potential for coadjoint orbit of $G \ltimes \mathfrak{g}^*$, the kinetic term we propose for the distinguished puncture is obtained from the canonical symplectic potential on the cotangent bundle of $G \ltimes \mathfrak{g}^*$. Both these symplectic potentials are discussed and derived in the Appendix A.1.

The full action for Chern-Simons theory on $\mathbb{R} \times S_{g,n}^\infty$ then depends on the gauge field A, the group valued functions h_i, g of x^0 and the Lie algebra valued function T of x^0 and is given by

$$\begin{aligned} &S[A_S, A_0, h_i, T, g] = \\ &\quad \int_{\mathbb{R}} dx^0 \int_{S_{g,n}^\infty} \tfrac{k}{4\pi} \langle \partial_0 A_S \wedge A_S \rangle - \int_{\mathbb{R}} dx^0 \sum_{i=1}^n \langle D_i \, , \, h_i^{-1} \partial_0 h_i \rangle + \int_{\mathbb{R}} dx^0 \langle T, g \partial_0 g^{-1} \rangle \\ &+ \int_{\mathbb{R}} dx^0 \int_{S_{g,n}^\infty} \langle A_0 \, , \, \tfrac{k}{2\pi} F_S - T \delta^{(2)}(x - x_\infty) dx^1 \wedge dx^2 - \sum_{i=1}^n T_i \delta^{(2)}(x - x_{(i)}) dx^1 \wedge dx^2 \rangle. \end{aligned} \tag{3.4}$$

Variation of the action (3.4) yields the equations of motion and constraints. Varying with respect to the Lagrange multiplier A_0 we obtain the constraint

$$\tfrac{k}{2\pi}F_S(x) = T\delta^{(2)}(x - x_\infty)dx^1 \wedge dx^2 + \sum_{i=1}^{n} T_i\delta^{(2)}(x - x_{(i)})dx^1 \wedge dx^2, \tag{3.5}$$

which imposes the required delta-function singularities of the curvature at the punctures as well as the vanishing of the curvature F_S outside the punctures. Variation with respect to the spatial component A_S of the gauge field gives

$$\partial_0 A_S = d_S A_0 + [A_S, A_0]. \tag{3.6}$$

Combined with (3.5) and the decomposition (2.3) this equation implies that the three-dimensional curvature F (2.2) on $\mathbb{R} \times S_{g,n}^{\infty}$ vanishes outside the world lines of the punctures. Varying with respect to h_i we obtain

$$\partial_0 T_i(x^0) = [T_i(x^0), A_0(x^0, x_{(i)})], \tag{3.7}$$

and variation of T gives

$$A_0(x^0, x_\infty) = g\partial_0 g^{-1}(x^0). \tag{3.8}$$

Finally, varying g yields

$$\partial_0 T = [T, g\partial_0 g^{-1}]. \tag{3.9}$$

Note that the evolution of both T and the $T_{(i)}$ is by conjugation. In particular, regularity is therefore preserved under evolution.

The action is invariant under gauge transformations which combine the usual gauge transformations of the gauge field with transformations of the variables associated to the punctures. Let $\gamma : \mathbb{R} \times S_{g,n}^{\infty} \to G \ltimes \mathfrak{g}^*$ be an arbitrary smooth function which is also well defined on the world lines of the punctures. Then the action is invariant under

$$\begin{aligned} &A_S \mapsto \gamma A_S \gamma^{-1} + \gamma d_S \gamma^{-1} \qquad && A_0 \mapsto \gamma A_0 \gamma^{-1} + \gamma \partial_0 \gamma^{-1} \qquad && h_i \mapsto \gamma(x^0, x_{(i)})h_i \\ &T \mapsto \gamma(x^0, x_\infty)T\gamma^{-1}(x^0, x_\infty) && g \mapsto \gamma(x^0, x_\infty)g. && \end{aligned} \tag{3.10}$$

In addition, the action (3.4) is invariant (up to a total x^0 derivative) under the transformations

$$h_i \mapsto h_i c_i \tag{3.11}$$

with functions $c_i : \mathbb{R} \to G \ltimes \mathfrak{g}^*$ taking values in the stabiliser group of D_i, i.e. the subgroup of elements of $G \ltimes \mathfrak{g}^*$ whose coadjoint action leaves D_i invariant. The gauge transformation (3.11) arises because of the redundancy in the parametrisation (3.2) of T_i via h_i.

For our calculation of the symplectic structure in the next section we need to parametrise the Lie algebra element T in analogy with (3.2) in terms of a general group element $h \in G \ltimes \mathfrak{g}^*$

and an element D in one of the Cartan subalgebras of $\mathfrak{g}\tilde{\oplus}\mathfrak{g}^*$. As explained in Sect. 2, this is equivalent to assuming that T is a regular element of $\mathfrak{g}\tilde{\oplus}\mathfrak{g}^*$. Regularity is preserved under conjugation and, following the remark made after (3.9), the assumption of regularity is thus consistent with the equations of motion for T. We assume moreover that T can be conjugated into an abelian Cartan subalgebra. As explained in Sect. 2, this assumption is automatically fulfilled for regular elements when G is semi-simple. We pick one such abelian Cartan subalgebra, denoted $\mathfrak{c}$ as in Sect. 2, and write T as

$$T = hDh^{-1} \tag{3.12}$$

with a phase space variable $D \in \mathfrak{c}$. Note that choices of non-conjugate Cartan subalgebras lead, in general, to different theories. Also, we observe that in trading the dynamical variable T for the two dynamical variables h and D we have introduced a redundancy which leads to an additional gauge invariance, as we shall see.

Using the parametrisation (3.12) we rewrite the kinetic term of the distinguished puncture in (3.4) as

$$\langle T, g\partial_0 g^{-1}\rangle = \langle D, w^{-1}\partial_0 w\rangle - \langle D, h^{-1}\partial_0 h\rangle \tag{3.13}$$

with

$$w = g^{-1}h, \tag{3.14}$$

see also [13] where a similar coordinate transformation is discussed. The action (3.4) then becomes

$$\begin{aligned} &S[A_S, A_0, h_i, D, h, w] = \\ &\quad \int_{\mathbb{R}} dx^0 \int_{S^\infty_{g,n}} \tfrac{k}{4\pi}\langle \partial_0 A_S \wedge A_S\rangle - \int_{\mathbb{R}} dx^0 \sum_{i=1}^n \langle D_i\, ,\, h_i^{-1}\partial_0 h_i\rangle - \int_{\mathbb{R}} dx^0 \langle D, h^{-1}\partial_0 h\rangle \\ &\quad + \int_{\mathbb{R}} dx^0 \int_{S^\infty_{g,n}} \langle A_0\, ,\, \tfrac{k}{2\pi}F_S - hDh^{-1}\delta^{(2)}(x - x_\infty)dx^1\wedge dx^2 - \sum_{i=1}^n T_i \delta^{(2)}(x - x_{(i)})dx^1 \wedge dx^2\rangle \\ &\quad + \int_{\mathbb{R}} dx^0 \langle D, w^{-1}\partial_0 w\rangle. \end{aligned} \tag{3.15}$$

Expressed in the variables h, w and D the gauge transformation (3.10) reads

$$\begin{array}{lll} A_S \mapsto \gamma A_S \gamma^{-1} + \gamma d\gamma^{-1} & A_0 \mapsto \gamma A_0\gamma^{-1} + \gamma\partial_0\gamma^{-1} & h_i \mapsto \gamma(x^0, x_{(i)})h_i \\ h \mapsto \gamma(x^0, x_\infty)h & D \mapsto D & w \mapsto w. \end{array} \tag{3.16}$$

Furthermore, we have the anticipated additional gauge invariance reflecting the redundancy of the new variables D, h, w: given $g \in G \ltimes \mathfrak{g}^*$ and $T \in \mathfrak{g}\tilde{\oplus}\mathfrak{g}^*$ the relations (3.12) and (3.14) only define $D \in C, h, w \in G \ltimes \mathfrak{g}^*$ up to simultaneous right-multiplication of h and w by an element of the stabiliser group of D. As a result the action (3.15) is invariant under

$$h \mapsto hc \qquad w \mapsto wc \tag{3.17}$$

with a x^0-dependent function $c : \mathbb{R} \to G \ltimes \mathfrak{g}^*$ which, for each x^0, takes values in the stabiliser group of $D(x^0)$.

4 Combinatorial description of the symplectic structure

4.1 Outline of the method

In this section, we derive a description of the symplectic structure on the phase space in terms of a finite number of variables closely related to the holonomies of certain curves on the surface $S^\infty_{g,n}$. We start with the canonical symplectic form associated to the action (3.15)

$$\Omega = -\,\delta\langle D, w^{-1}\delta w\rangle + \delta\langle D, h^{-1}\delta h\rangle + \sum_{i=1}^{n} \delta\langle D_i, h_i^{-1}\delta h_i\rangle + \frac{k}{4\pi}\int_{S^\infty_{g,n}} \langle \delta A_S \wedge \delta A_S\rangle, \qquad (4.1)$$

where the symbol δ stands for exterior differentiation in field space. The form (4.1) is symplectic (non-degenerate) on the enlarged phase space parametrised by the variables $D \in \mathfrak{c}$, $D_i \in \cup_\iota \mathfrak{c}_\iota$, $w, h, h_i \in G \ltimes \mathfrak{g}^*$ and the spatial gauge field A_S, where the conditions that the D_i are fixed and that A_S satisfies (3.5) are not imposed. Both of these conditions are first class constraints, so to obtain the physical phase space we need to impose the constraints and divide the phase space by the gauge transformations generated by these constraints. In addition, we have to divide by the gauge transformations (3.17) arising from the redundant parametrisation (3.12). The form Ω then determines a unique symplectic structure on the physical phase space.

In practice it is often difficult or impossible to write down an explicit formula for the symplectic structure on the physical phase space. However, in many situations it is possible to describe the symplectic structure in terms of a two-form Ω on an auxiliary space, in the following referred to as extended phase space and denoted by $\mathcal{P}_{ext}$, with some (preferably most) but not all of the gauge freedom divided out. The remaining gauge freedom is encoded in a set of constraints defining a constraint surface $\mathcal{C} \subset \mathcal{P}_{ext}$, on which this two-form Ω agrees with the pull-back of the symplectic structure on the physical phase space.

For Chern-Simons theory on a spatial surface $S_{g,n}$ of genus g and with n ordinary punctures, such a description has been achieved by Alekseev and Malkin [8]. They consider the case of a simple, complex gauge group H with fixed H-conjugacy classes associated to each puncture. In their description, the phase space is parametrised in terms of a finite set of variables linked to the holonomies along the generators of the surface's fundamental group, and its symplectic structure is given in terms of its pull-back to the manifold $H^{n+2g} \times C^{n+g}$, where $C \subset H$ is a fixed Cartan subgroup.

In our derivation of the symplectic structure defined by (4.1) we closely follow the method introduced by Alekseev and Malkin [8]. However, we extend their formalism to deal with the distinguished puncture at x_∞ where the curvature is not restricted to lie in a fixed conjugacy class. The resulting extended phase space $\mathcal{P}_{ext}$ is the manifold $(G \ltimes \mathfrak{g}^*)^{n+2g+1} \times \mathfrak{c}$, where the first $n + 2g$ copies of $G \ltimes \mathfrak{g}^*$ correspond the generators of the surface's $S^\infty_{g,n}$ fundamental group, the last copy stands for the element $w \in G \ltimes \mathfrak{g}^*$ and $\mathfrak{c}$ parametrises the element D in (4.1). The symplectic form Ω determines a two-form on this space which we also denote by Ω, and for which we derive an explicit formula.

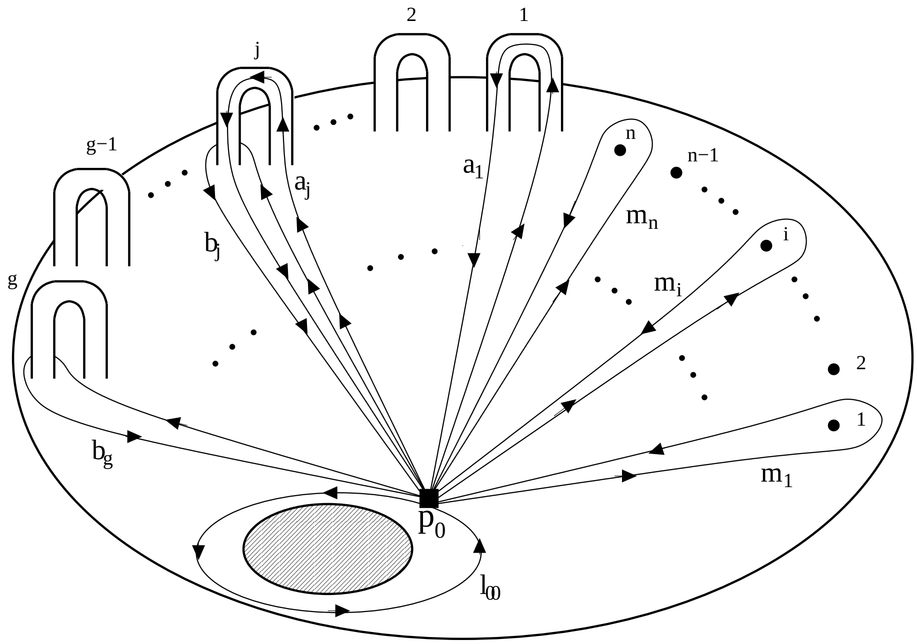


Figure 1: The generators of the fundamental group $\pi_1(S^{\infty}_{g,n}, p_0)$

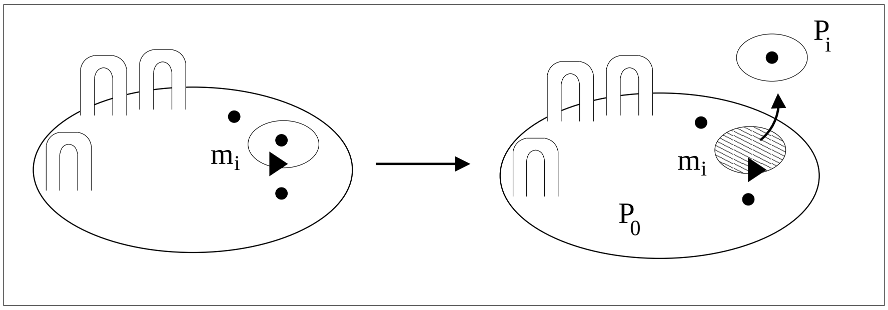
As the derivation of this two-form is both lengthy and technical, we start by outlining the main steps in the method introduced in [8]. The first step is to cut the spatial surface $S^{\infty}_{g,n}$ along a set of generators of its fundamental group, which results in a simply connected polygon and $n+1$ punctured discs. The second step is to trivialise the gauge field on the polygon and to decompose it into a regular and a singular component on the punctured discs. The integral in (3.15) can then be transformed into a set of boundary integrals along each cut. Finally, one relates the boundary integrals on the two sides of each cut by means of a continuity condition on the gauge field. After evaluating and summing the two boundary integrals in this way, one can explicitly perform the integration and express the result in terms of the holonomies along the generators of the fundamental group.

4.2 Cutting the surface

We begin by picking a point p_0 on the surface, distinct from the marked points $x_{(i)}$ and x_∞, and loops a_j, b_j, m_i, $j = 1, \ldots, g$, $i = 1, \ldots, n$, and l_∞, all based at p_0, whose homotopy classes generate the surface's fundamental group. There are two curves a_j and b_j for each handle, one loop m_i for each of the ordinary punctures and finally one loop l_∞ encircling the distinguished puncture, see Fig. 1. The fundamental group $\pi_1(S^{\infty}_{g,n}, p_0)$ is generated by the homotopy classes of a_j, b_j, m_i, $j = 1, \ldots, g$,$i = 1, \ldots, n$, and l_∞, subject to the relation

$$l_\infty [b_n, a_n^{-1}] \cdots [b_1, a_1^{-1}] m_n \cdots m_1 = 1. \tag{4.2}$$

We denote the A_S-holonomies along the generators by the corresponding capital letters, so M_i is the holonomy around m_i, A_j and B_j are the holonomies around the a- and b-cycle of the j-th handle, and L_∞ is the holonomy around the distinguished puncture.



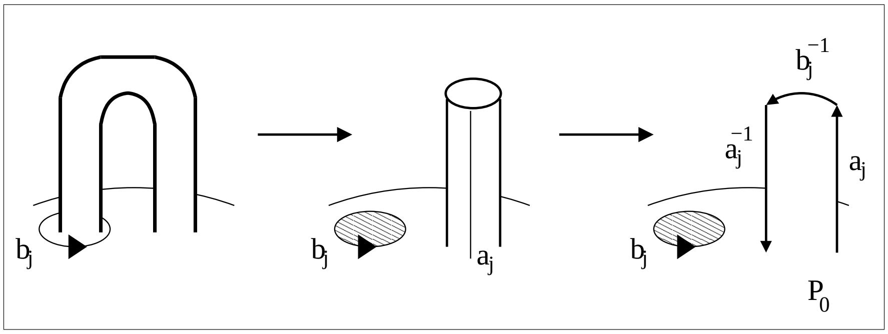


Figure 2: The cutting procedure for punctures and handles

The next step is to cut the surface $S^{\infty}_{g,n}$ along the curves a_j, b_j, m_i, $j = 1, \ldots, g$, $i = 1, \ldots, n$, and l_∞, as shown in Fig. 2. This results in $n+2$ disconnected regions. Of these, n are punctured discs P_i surrounding the ordinary punctures, and one is a punctured disc P_∞ surrounding the distinguished puncture. The final piece is a simply connected polygon P_0, shown in figure Fig. 3.

The integral in (4.1) can now be written as sum of contributions from each of the regions:

$$\begin{aligned} &\frac{k}{4\pi}\int_{S^{\infty}_{g,n}} \langle \delta A_S \wedge \delta A_S \rangle \qquad (4.3) \\ = \; &\frac{k}{4\pi}\int_{P_0} \langle \delta A_S \wedge \delta A_S \rangle + \frac{k}{4\pi}\int_{P_\infty} \langle \delta A_S \wedge \delta A_S \rangle + \frac{k}{4\pi}\sum_{i=1}^{n}\int_{P_i} \langle \delta A_S \wedge \delta A_S \rangle . \end{aligned}$$

Still following Alekseev and Malkin, we now show how to convert each of the above integrals into boundary integrals and how to express the boundary integrals in terms of $G \ltimes \mathfrak{g}^*$-elements parametrising the holonomies M_i, A_j, B_j and L_∞.

4.3 Transformation into boundary integrals

Since the interior of the polygon P_0 is simply connected, one can express the flat gauge field A_S on P_0 as a pure gauge

$$A_S|_{P_0} = \gamma_0 d\gamma_0^{-1} \qquad \text{with } \gamma_0(p_0) = 1, \qquad (4.4)$$

for some $G \ltimes \mathfrak{g}^*$-valued function γ_0 on P_0. The condition that γ_0 is the identity at the base point p_0 is a partial gauge fixing. The punctured discs P_i, $i = 1, \ldots, n$, and P_∞ are not

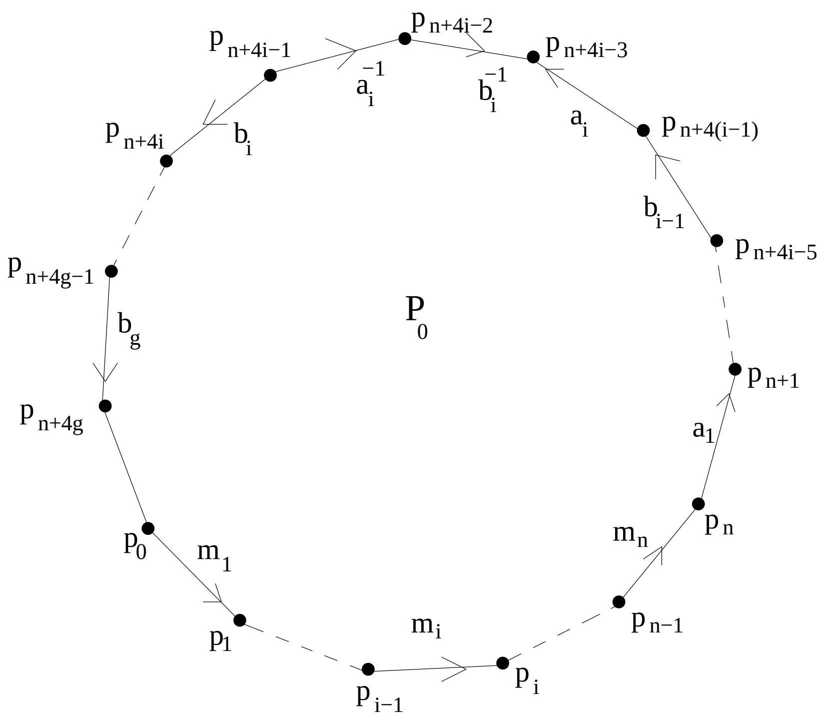


Figure 3: The polygon P_0

simply connected, and therefore it is not possible to write A_S as a pure gauge here. As explained in [8] one can, however, introduce angular coordinates ϕ_i, $i = 1, \ldots, n$, and ϕ, each with range $[0, 2\pi]$, in the discs around the ordinary punctures and the distinguished puncture, and give an explicit formula for a gauge field which is flat on the punctured disc but has the desired curvature singularity at the puncture. On each of the discs P_i such a gauge field is

$$B_i = \tfrac{1}{k} D_i d\phi_i, \tag{4.5}$$

where D_i is the fixed element in $\cup_{\iota\in I}\mathfrak{c}_\iota$ associated to the i-the puncture. We parametrise this element explicitly in terms of $\mathfrak{g}$- and $\mathfrak{g}^*$ elements as

$$D_i = \tfrac{k}{2\pi}(\boldsymbol{\mu}_i, \boldsymbol{s}_i). \tag{4.6}$$

On the disc P_∞ a flat gauge field with a curvature singularity at the distinguished puncture is given by

$$B = \tfrac{1}{k} D d\phi, \tag{4.7}$$

where D is an arbitrary element in the Cartan subalgebra $\mathfrak{c}$, which we parametrise as

$$D = -\tfrac{k}{2\pi}(\boldsymbol{\mu}, \boldsymbol{s}). \tag{4.8}$$

The reason for adopting a different sign convention here compared to (4.6) will be become clear later. The general form of the gauge field on the punctured discs is obtained by applying

a gauge transformation to the singular gauge fields (4.5) and (4.7). On P_i this gives

$$A_S|_{P_{M_i}} = \tfrac{1}{k}\gamma_{M_i} D_i d\phi_i \gamma_{M_i}^{-1} + \gamma_{M_i} d\gamma_{M_i}^{-1} \quad \text{with} \quad \gamma_{M_i}(x_{(i)}) = h_i, \tag{4.9}$$

where the condition $\gamma_{M_i}(x_{(i)}) = h_i$ ensures that the constraint (3.5) is satisfied. Similarly, on the punctured disc P_∞ we can write the general gauge potential as

$$A_S|_{P_\infty} = \tfrac{1}{k}\gamma_\infty D d\phi \gamma_\infty^{-1} + \gamma_\infty d\gamma_\infty^{-1} \quad \text{with} \quad \gamma_\infty(x_\infty) = h. \tag{4.10}$$

Using the expressions (4.4) and (4.9) for the gauge field, Alekseev and Malkin showed in [8] how to to transform the integrals over P_0 and P_i, $i = 1, \ldots, n$ in (4.3) into boundary integrals. The main tool is a technical lemma, summarised as Lemma B.1 in Appendix B.

Application of this lemma to the region P_0 gives

$$\frac{k}{4\pi}\int_{P_0} \langle \delta A_S \wedge \delta A_S \rangle = \frac{k}{4\pi}\int_{\partial P_0} \langle \delta\gamma_0^{-1}\gamma_0 d\left(\delta\gamma_0^{-1}\gamma_0\right)\rangle, \tag{4.11}$$

and the result for the punctured discs P_i $i = 1, \ldots, n$ is [8]

$$\begin{aligned} &\frac{k}{4\pi}\int_{P_i} \langle \delta A_S \wedge \delta A_S \rangle \qquad\qquad (4.12)\\ = \quad & -\delta\langle D_i \,,\, h_i^{-1}\delta h_i\rangle + \frac{k}{4\pi}\int_{\partial P_i} \langle \delta\gamma_{M_i}^{-1}\gamma_{M_i} d\left(\delta\gamma_{M_i}^{-1}\gamma_{M_i}\right)\rangle - \tfrac{2}{k}\delta\langle D_i \delta\gamma_{M_i}^{-1}\gamma_{M_i}\rangle d\phi_i. \end{aligned}$$

To find the corresponding result for the punctured disc P_∞, we use Lemma B.1 with $B = \frac{1}{k}D d\phi$, $\gamma = \gamma_\infty$. The term $\langle \delta B \wedge \delta B\rangle$ vanishes because D is in the abelian Cartan subalgebra $\mathfrak{c}$. The resulting contribution to the symplectic structure is

$$\begin{aligned} &\frac{k}{4\pi}\int_{P_\infty} \langle \delta A_S \wedge \delta A_S \rangle \qquad\qquad (4.13)\\ = \quad & -\delta\langle D \,,\, h^{-1}\delta h\rangle + \frac{k}{4\pi}\int_{\partial P_\infty} \langle \delta\gamma_\infty^{-1}\gamma_\infty d\left(\delta\gamma_\infty^{-1}\gamma_\infty\right)\rangle - \tfrac{2}{k}\delta\langle D \delta\gamma_\infty^{-1}\gamma_\infty\rangle d\phi, \end{aligned}$$

which is formally the same as the contribution for the punctured discs P_i. However, for further calculations we need to keep in mind that $\delta D \neq 0$.

Collecting the expressions for the integrals over the regions P_0, P_i, P_∞ and inserting them into (4.1), we find that the first terms in (4.12) and (4.13) are cancelled by terms in (4.1). Thus the two-form (4.1) is given by

$$\begin{aligned} \Omega = &- \delta\langle D \,,\, w^{-1}\delta w\rangle + \frac{k}{4\pi}\int_{\partial P_\infty} \langle \delta\gamma_\infty^{-1}\gamma_\infty d\left(\delta\gamma_\infty^{-1}\gamma_\infty\right)\rangle - \tfrac{2}{k}\delta\langle D \delta\gamma_\infty^{-1}\gamma_\infty\rangle d\phi \qquad (4.14)\\ &+ \frac{k}{4\pi}\int_{\partial P_0} \langle \delta\gamma_0^{-1}\gamma_0 d\left(\delta\gamma_0^{-1}\gamma_0\right)\rangle + \frac{k}{4\pi}\sum_{i=1}^{n}\int_{\partial P_i} \langle \delta\gamma_{M_i}^{-1}\gamma_{M_i} d\left(\delta\gamma_{M_i}^{-1}\gamma_{M_i}\right)\rangle - \tfrac{2}{k}\delta\langle D_i \delta\gamma_{M_i}^{-1}\gamma_{M_i}\rangle d\phi_i. \end{aligned}$$

The boundary integrals in (4.14) involve exactly two integrations along each cut, one from each side. The next step in the evaluation is the summation over these two contributions

by means of an overlap condition. The gauge potential is required to be smooth, but this requirement only determines the trivialising gauge transformation up to a constant element in the gauge group $G \ltimes \mathfrak{g}^*$. In the next subsection we explain, following [8], how to relate these constant elements to the holonomies of the connection A_S around the generators of the fundamental group of $S^\infty_{g,n}$.

4.4 Overlap conditions

The boundary of the polygon P_0 has the decomposition

$$\partial P_0 = l_\infty \ \cup \bigcup_{i=1}^{n} m_i \bigcup_{i=1}^{g} \left(a_i \cup b_i \cup a_i^{-1} \cup b_i^{-1}\right). \tag{4.15}$$

For the cuts along the generators $m_1, \ldots, m_n$ the continuity of the gauge field amounts to the condition

$$\gamma_0 d\gamma_0^{-1}|_{m_i} = \left(\tfrac{1}{k}\gamma_{M_i} D_i d\phi_i \gamma_{M_i}^{-1} + \gamma_{M_i} d\gamma_{M_i}^{-1}\right)|_{m_i}, \tag{4.16}$$

which can be expressed equivalently as

$$\gamma_0^{-1}|_{m_i} = N_{M_i} D_{M_i}(\phi_i)\gamma_{M_i}^{-1}|_{m_i} \quad \text{with} \quad dN_{M_i} = 0, \tag{4.17}$$

where $C_{M_i}(\phi_i) = \exp(\frac{1}{k}D_i\phi_i)$ and N_{M_i} is an arbitrary but constant element of the gauge group $G \ltimes \mathfrak{g}^*$. For the cuts along $a_1, b_1, \ldots, a_g, b_g$ the continuity condition is

$$\gamma_0' d\left(\gamma_0'\right)^{-1}|_x = \gamma_0'' d\left(\gamma_0''\right)^{-1}|_x, \tag{4.18}$$

where $x = a_1, b_1, \ldots, a_g, b_g$ and γ_0', γ_0'' denote the values of the trivialising gauge transformation γ_0 at the two sides of the polygon P_0 associated to the curve x. The condition (4.18) is equivalent to

$$\left(\gamma_0'\right)^{-1}|_x = N_x \left(\gamma_0''\right)^{-1}|_x \quad \text{with} \quad dN_x = 0. \tag{4.19}$$

Finally, for the cut along the curve l_∞ we have

$$\gamma_0 d\gamma_0^{-1}|_{l_\infty} = \left(\tfrac{1}{k}\gamma_\infty D d\phi \gamma_\infty^{-1} + \gamma_\infty d\gamma_\infty^{-1}\right)|_{l_\infty} \tag{4.20}$$

or, equivalently,

$$\gamma_0^{-1}|_{l_\infty} = N_\infty C_\infty(\phi)\gamma_\infty^{-1}|_{l_\infty} \quad \text{with} \quad dN_\infty = 0, \tag{4.21}$$

where $C_\infty(\phi) = \exp(\frac{1}{k}D\phi)$.

The values of γ_0, γ_{M_i} and γ_∞ at the endpoints of the cuts are related to the holonomies of the generators of $\pi_1(S^\infty_{g,n}, p_0)$. We start by considering the polygon P_0. Denoting the endpoints of the cuts by p_i, $i = 0, \ldots, n+4g$, as shown in Fig. 2, so that in particular m_i runs from

p_{i-1} to p_i and l_∞ from p_{n+4g} to p_0, one finds that the parallel transport along the cut with endpoints p_{i-1} and p_i is given by

$$PT_{p_{i-1}\to p_i} = \gamma_0(p_i)\gamma_0^{-1}(p_{i-1}). \tag{4.22}$$

As mentioned after (4.4), one can set $\gamma_0(p_0) = 1$, thus partially fixing the gauge. Then the values of γ_0 at the endpoints of the cuts are related to the holonomies A_j, B_j, M_i, $j = 1, \ldots, g$, $i = 1, \ldots, n$, and L_∞ by the following equations:

$$\begin{aligned}
&\gamma_0(p_i) = K_i^{-1} := M_i \cdots M_1 \qquad \text{for } 1 \le i \le n \\
&\gamma_0(p_{n+4j-3}) = K_{n+2j-1}^{-1} := A_j[B_{j-1}, A_{j-1}^{-1}] \cdots [B_1, A_1^{-1}]M_n \cdots M_1 \\
&\gamma_0(p_{n+4j-2}) = B_j^{-1}A_j[B_{j-1}, A_{j-1}^{-1}] \cdots [B_1, A_1^{-1}]M_n \cdots M_1 \\
&\gamma_0(p_{n+4j-1}) = A_j^{-1}B_j^{-1}A_j[B_{j-1}, A_{j-1}^{-1}] \cdots [B_1, A_1^{-1}]M_n \cdots M_1 \\
&\gamma_0(p_{n+4j}) = K_{n+2j}^{-1} := [B_j, A_j{}^{-1}] \cdots [B_1, A_1^{-1}]M_n \cdots M_1 \quad \text{for } 1 \le j \le g.
\end{aligned} \tag{4.23}$$

Alternatively, considering the parallel transport from p_0 to p_{n+4g} we can write the condition for p_{n+4g} in terms of L_∞, using (4.22) and $\gamma(p_0) = 1$:

$$\gamma_0(p_{n+4g}) = L_\infty^{-1}, \tag{4.24}$$

so that - in agreement with (4.2) - the holonomy L_∞ is given by

$$L_\infty = \left([B_n, A_n^{-1}] \cdots [B_1, A_1^{-1}]M_n \cdots M_1\right)^{-1}. \tag{4.25}$$

Note that the overlap conditions (4.18) ensure that the holonomies along the two sides of P_0 corresponding to each a- and b-cycle are indeed the same

$$\begin{aligned}
A_i &= \gamma_0(p_{n+4i-3})\gamma_0^{-1}(p_{n+4(i-1)}) = \gamma_0(p_{n+4i-2})\gamma_0^{-1}(p_{n+4i-1}) \\
B_i &= \gamma_0(p_{n+4i})\gamma_0^{-1}(p_{n+4i-1}) = \gamma_0(p_{n+4i-3})\gamma_0^{-1}(p_{n+4i-2}).
\end{aligned} \tag{4.26}$$

We now consider the punctured discs P_i for the ordinary punctures. The parallel transport around P_i gives the holonomy M_i, which we can now relate to the parametrisation of the gauge field $A_S|_{m_i}$ in (4.9). The gauge transformation γ_{M_i} is single-valued, so $\gamma_{M_i}(p_{i-1}) = \gamma_{M_i}(p_i)$ and therefore

$$M_i = g_i C_i^{-1} g_i^{-1} \qquad \text{with} \qquad C_i = \exp(\tfrac{2\pi}{k}D_i) = (e^{\boldsymbol{\mu}_i}, \boldsymbol{s}_i),\ g_i = \gamma_{M_i}(p_i). \tag{4.27}$$

Similarly, on the punctured disc P_∞ we have the expression (4.10) with $\gamma_\infty(p_0) = \gamma_\infty(p_{n+4g})$. Recalling the parametrisation of D in (4.8) we have

$$L_\infty = K_{n+2g} = g_\infty C^{-1} g_\infty^{-1}, \quad \text{with} \quad C = \exp(\tfrac{2\pi}{k}D) = (e^{-\boldsymbol{\mu}}, -\boldsymbol{s}),\ g_\infty = \gamma_\infty(p_0). \tag{4.28}$$

4.5 Evaluation in terms of holonomies

We will now combine the two integrals along each cut and express the two-form (4.14) in terms of the holonomies M_i, A_j, B_j, L_∞. We begin with the contributions of all cuts except the one around the distinguished puncture:

$$\begin{aligned}\Omega_b :=& \frac{k}{4\pi}\int_{\partial P_0 - l_\infty} \langle \delta\gamma_0^{-1}\gamma_0 d\left(\delta\gamma_0^{-1}\gamma_0\right)\rangle \\ &+\frac{k}{4\pi}\sum_{i=1}^n \int_{\partial P_i} \langle \delta\gamma_{M_i}^{-1}\gamma_{M_i} d\left(\delta\gamma_{M_i}^{-1}\gamma_{M_i}\right)\rangle - \tfrac{2}{k}\delta\langle D_i \delta\gamma_{M_i}^{-1}\gamma_{M_i}\rangle d\phi_i.\end{aligned} \tag{4.29}$$

For the case where the gauge group H is a simple complex group or its compact real form a formula for Ω_b was given by Alekseev and Malkin in [8] as a sum of the contributions Ω_{M_i}, $i = 1,\ldots,n$, from the ordinary punctures (B.7) and the contributions Ω_{H_j}, $i = 1,\ldots,g$, from the handles (B.15). In Appendix B we quote their expressions for Ω_{M_i} and Ω_{H_j}, sketch their derivation, and also explain modifications required for our gauge groups $G \ltimes \mathfrak{g}^*$, see Lemma B.2, Lemma B.3 and Lemma B.4. Summing over the contributions Ω_{M_i}, Ω_{H_j} for each ordinary puncture and handle, one finds

$$\begin{aligned}\Omega_b =& \sum_{i=1}^n \Omega_{M_i} + \sum_{i=1}^g \Omega_{H_i} \\ =& \frac{k}{4\pi}\sum_{i=1}^n \langle C_i g_i^{-1}\delta g_i C_i^{-1} \wedge g_i^{-1}\delta g_i\rangle - \frac{k}{4\pi}\sum_{i=1}^{n+2g}\langle \delta K_i K_i^{-1} \wedge \delta K_{i-1}K_{i-1}^{-1}\rangle \\ &- \frac{k}{4\pi}\sum_{i=1}^g \left(\langle A_i^{-1}\delta A_i \wedge B_i^{-1}\delta B_i\rangle + \langle \delta(B_i A_i B_i^{-1}) B_i A_i^{-1} B_i^{-1} \wedge \delta B_i\, B_i^{-1}\rangle\right).\end{aligned} \tag{4.30}$$

It remains to evaluate the contribution to the two-form (4.14) from the cut l_∞ around the distinguished puncture, which is given by

$$\begin{aligned}\Omega_\infty : \;=\; & -\delta\langle D\,,\, w^{-1}\delta w\rangle + \frac{k}{4\pi}\int_{\partial P_0 \cap \partial P_\infty} \langle \delta\gamma_0^{-1}\gamma_0 d(\delta\gamma_0^{-1}\gamma_0)\rangle \\ & + \frac{k}{4\pi}\int_{\partial P_\infty} \langle \delta\gamma_\infty^{-1}\gamma_\infty d\left(\delta\gamma_\infty^{-1}\gamma_\infty\right)\rangle - \tfrac{2}{k}\delta\langle D\delta\gamma_\infty^{-1}\gamma_\infty\rangle d\phi.\end{aligned} \tag{4.31}$$

The overlap condition (4.21) implies

$$\begin{aligned}\langle \delta\gamma_0^{-1}\gamma_0 d(\delta\gamma_0^{-1}\gamma_0)\rangle =& \langle \delta\gamma_\infty^{-1}\gamma_\infty d(\delta\gamma_\infty^{-1}\gamma_\infty)\rangle - 2\delta\langle C_\infty^{-1} dC_\infty \delta\gamma_\infty^{-1}\gamma_\infty\rangle \\ & + d\langle \delta N_\infty N_\infty^{-1}\delta\gamma_0^{-1}\gamma_0\rangle + d\langle C_\infty^{-1}\delta C_\infty \delta\gamma_\infty^{-1}\gamma_\infty\rangle,\end{aligned} \tag{4.32}$$

and we obtain

$$\Omega_\infty = -\delta\langle D, w^{-1}\delta w\rangle + \tfrac{k}{4\pi}\langle \delta N_\infty N_\infty^{-1}\delta\gamma_0^{-1}\gamma_0\rangle|_{p_{n+4g}}^{p_0} + \tfrac{k}{4\pi}\langle C_\infty^{-1}\delta C_\infty, \delta\gamma_\infty^{-1}\gamma_\infty\rangle|_{p_{n+4g}}^{p_0}. \tag{4.33}$$

Recalling the notation $g_\infty = \gamma_\infty(p_0) = \gamma_\infty(p_{n+4g})$ from (4.28) and $C_\infty(\phi) = \exp(\frac{1}{k}D\phi)$ (4.21), we have for the last term in (4.33)

$$\begin{aligned}\tfrac{k}{4\pi}\langle C_\infty^{-1}\delta C_\infty, \delta\gamma_\infty^{-1}\gamma_\infty\rangle|_{p_{n+4g}}^{p_0} &= -\tfrac{1}{4\pi}(\phi(p_0)-\phi(p_{n+4g}))\langle\delta D, g_\infty^{-1}\delta g_\infty\rangle \\ &= -\tfrac{1}{2}\langle\delta D\,,\, g_\infty^{-1}\delta g_\infty\rangle \\ &= \tfrac{k}{4\pi}\langle(\delta\boldsymbol{\mu},\delta\boldsymbol{s})\,,\, g_\infty^{-1}\delta g_\infty\rangle,\end{aligned} \tag{4.34}$$

where the last equality follows from the definition of C in (4.28) and we used $\delta\phi(p_0) = \delta\phi(p_{n+4g}) = 0$, $\phi(p_0) - \phi(p_{n+4g}) = 2\pi$. To evaluate the second term in (4.33), we recall that $\gamma_0(p_{n+4g}) = K_{n+2g}^{-1}$ and that we imposed the condition $\gamma_0(p_0) = 1$. Together with (4.21), this implies $N_\infty = g_\infty e^{-\frac{1}{k}D\phi(p_0)}$ and, using (4.28), we find

$$\Omega_\infty = \tfrac{k}{2\pi}\delta\langle(\boldsymbol{\mu},\boldsymbol{s}), w^{-1}\delta w\rangle + \tfrac{k}{4\pi}\langle K_{n+2g}^{-1}\delta K_{n+2g}, \delta g_\infty g_\infty^{-1}\rangle + \tfrac{k}{4\pi}\langle(\delta\boldsymbol{\mu},\delta\boldsymbol{s})\,,\, g_\infty^{-1}\delta g_\infty\rangle. \tag{4.35}$$

Adding this expression to the contribution (4.30) from the cuts along the generators of the fundamental group, we find that the two-form (4.14) is given by

$$\begin{aligned}\Omega =& \frac{k}{4\pi}\sum_{i=1}^{n}\langle C_i g_i^{-1}\delta g_i C_i^{-1}\wedge g_i^{-1}\delta g_i\rangle - \frac{k}{4\pi}\sum_{i=1}^{n+2g}\langle\delta K_i K_i^{-1}\wedge\delta K_{i-1}K_{i-1}^{-1}\rangle \\ &-\frac{k}{4\pi}\sum_{i=1}^{g}\left(\langle A_i^{-1}\delta A_i\wedge B_i^{-1}\delta B_i\rangle + \langle\delta(B_iA_iB_i^{-1})B_iA_i^{-1}B_i^{-1}\wedge\delta B_i\,B_i^{-1}\rangle\right) \\ &+\frac{k}{2\pi}\delta\langle(\boldsymbol{\mu},\boldsymbol{s}), w^{-1}\delta w\rangle + \frac{k}{4\pi}\langle K_{n+2g}^{-1}\delta K_{n+2g}, \delta g_\infty g_\infty^{-1}\rangle + \frac{k}{4\pi}\langle(\delta\boldsymbol{\mu},\delta\boldsymbol{s})\,, g_\infty^{-1}\delta g_\infty\rangle.\end{aligned} \tag{4.36}$$

5 The symplectic structure for gauge groups $G \ltimes \mathfrak{g}^*$

Equation (4.36) is one of the main results of this paper. It gives an expression for the symplectic structure on the phase space in terms of the pull-back to the group $(G\ltimes\mathfrak{g}^*)^{n+2g+2}\times\mathfrak{c}$ in terms of the variables $g_1,\ldots,g_n$, $A_1,B_2,\ldots,A_g,B_g$, g_∞, w and $(\boldsymbol{\mu},\boldsymbol{s})$ and the constant elements $C_1,\ldots,C_n$. Although we have written this result using the explicit parametrisation of D in terms of the $\mathfrak{g}$-element $\boldsymbol{\mu}$ and the $\mathfrak{g}^*$-element $\boldsymbol{s}$, the result itself does not depend this parametrisation. When replacing $(\boldsymbol{\mu},\boldsymbol{s})$ by $-\frac{2\pi}{k}D$ the formula (4.36) holds for any Lie group H with an invariant pairing $\langle\,,\,\rangle$ on its Lie algebra.

In this section, we derive the other central result of our paper, the explicit evaluation of the two-form (4.36) for gauge groups of the form $G\ltimes\mathfrak{g}^*$. The calculations in this section make repeated use of two parametrisations of group elements in $G\ltimes\mathfrak{g}^*$. First, we write elements of $G\ltimes\mathfrak{g}^*$ in terms of an element $u\in G$ and an element $\boldsymbol{j}\in\mathfrak{g}^*$ as $(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j})$. As explained in Appendix A.2, this parametrisation is derived from a diffeormophism between $G\ltimes\mathfrak{g}^*$ and the dual Poisson-Lie group $G\times\mathfrak{g}^*$. The second parametrisation has its origins in dressing actions of $G\ltimes\mathfrak{g}^*$. As also explained in A.2 the dressing action of $(v,\boldsymbol{x})\in G\ltimes\mathfrak{g}^*$ on an element $(\tilde{u},-\tilde{\mathbf{j}})\in G\times\mathfrak{g}^*$ is simply conjugation of the corresponding element $(\tilde{u},-\mathrm{Ad}^*(\tilde{u}^{-1})\tilde{\mathbf{j}})$ in $G\ltimes\mathfrak{g}^*$. If $(\tilde{u},-\tilde{\mathbf{j}}) = (e^{-\boldsymbol{\mu}},-\boldsymbol{s})$ is in the abelian subgroup $T_\mathfrak{c}$ we obtain the following parametrisation

of elements in its dressing orbit:

$$(u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) = (v, \boldsymbol{x})(e^{-\boldsymbol{\mu}}, -\boldsymbol{s})(v, \boldsymbol{x})^{-1}. \tag{5.1}$$

The resulting expressions for u and $\boldsymbol{j}$ in terms of $(v, \boldsymbol{x})$ and $(e^{-\boldsymbol{\mu}}, -\boldsymbol{s})$ are given in (A.25).

We begin with a technical lemma concerning the parametrisation (5.1) which can be proved by direct calculation. Equipped with this lemma we then show how to write the two-form (4.36) as the exterior derivative of an explicitly given one-form.

Lemma 5.1 *For $K^{-1} = gC^{-1}g^{-1} = (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}) \in G \ltimes \mathfrak{g}^*$ with general elements $g = (v, \boldsymbol{x}) \in G \ltimes \mathfrak{g}^*$ and elements $C = (e^{\boldsymbol{\mu}}, \boldsymbol{s})$ in the abelian subgroup $T_{\mathfrak{c}}$, we have the following identity*

$$\begin{aligned}
&\langle Cg^{-1}\delta g C^{-1} \wedge g^{-1}\delta g\rangle - 2\langle \delta C C^{-1}\,,\, g^{-1}\delta g\rangle = -\langle K^{-1}\delta K\,,\delta g g^{-1}\rangle - \langle (\delta\boldsymbol{\mu}, \delta\boldsymbol{s})\,,\, g^{-1}\delta g\rangle \\
&= -\langle \delta\boldsymbol{j}\,,\, u^{-1}\delta u\rangle - \langle \delta\boldsymbol{s}\,,\, \delta\boldsymbol{\mu}\rangle - 2\delta\langle \boldsymbol{j}\,,\, \delta v v^{-1}\rangle + 2\delta\langle \boldsymbol{x}\,,\, v\delta\boldsymbol{\mu} v^{-1}\rangle.
\end{aligned} \tag{5.2}$$

Theorem 5.2 *If the holonomies around the generators of $\pi_1(S^\infty_{g,n}, p_0)$ are parametrised as*

$$\begin{aligned}
M_i &= (u_{M_i}, -\mathrm{Ad}^*(u_{M_i}^{-1})\boldsymbol{j}_{M_i}) = (v_{M_i}, \boldsymbol{x}_{M_i})(e^{-\boldsymbol{\mu}_i}, -\boldsymbol{s}_i)(v_{M_i}, \boldsymbol{x}_{M_i})^{-1} \qquad (5.3)\\
A_i &= (u_{A_i}, -\mathrm{Ad}^*(u_{A_i}^{-1})\boldsymbol{j}_{A_i})\\
B_i &= (u_{B_i}, -\mathrm{Ad}^*(u_{B_i}^{-1})\boldsymbol{j}_{B_i}),
\end{aligned}$$

with the inverse holonomy of the curve l_∞ around the distinguished puncture written as

$$\begin{aligned}
L_\infty^{-1} =& K_{n+2g}^{-1} = (u_{K_{n+2g}}, -\mathrm{Ad}^*(u_{K_{n+2g}}^{-1})\boldsymbol{j}_{K_{n+2g}}) = [B_g, A_g^{-1}]\cdots[B_1, A_1^{-1}]M_n\cdots M_1 \\
=&(v_\infty, \boldsymbol{x}_\infty)(e^{-\boldsymbol{\mu}}, -\boldsymbol{s})(v_\infty, \boldsymbol{x}_\infty)^{-1}
\end{aligned} \tag{5.4}$$

the two-form (4.36) is given by

$$\Omega = \frac{k}{2\pi}\delta\left(\langle(\boldsymbol{\mu}, \boldsymbol{s}), w^{-1}\delta w\rangle + \Theta + \langle \boldsymbol{j}_{K_{n+2g}}, \delta v_\infty v_\infty^{-1}\rangle - \langle \mathrm{Ad}^*(v_\infty)\boldsymbol{x}_\infty - \tfrac{1}{2}\boldsymbol{s}\,,\, \delta\boldsymbol{\mu}\rangle\right) \tag{5.5}$$

with the one-form

$$\begin{aligned}
\Theta =& \sum_{i=1}^{n}\langle\, \delta(u_{M_{i-1}}\cdots u_{M_1})(u_{M_{i-1}}\cdots u_{M_1})^{-1} - \delta v_{M_i} v_{M_i}^{-1}\,,\, \boldsymbol{j}_{M_i}\rangle \\
&+ \sum_{i=1}^{g}\langle\, \delta(u_{H_{i-1}}\cdots u_{M_1})(u_{H_{i-1}}\cdots u_{M_1})^{-1}\,,\, \boldsymbol{j}_{A_i}\rangle \\
&\qquad - \langle\, \delta(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}u_{H_{i-1}}\cdots u_{M_1})(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}u_{H_{i-1}}\cdots u_{M_1})^{-1}\,,\, \boldsymbol{j}_{A_i}\rangle \\
&+ \sum_{i=1}^{g}\langle\, \delta(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}u_{H_{i-1}}\cdots u_{M_1})(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}u_{H_{i-1}}\cdots u_{M_1})^{-1}\,,\, \boldsymbol{j}_{B_i}\rangle \\
&\qquad - \langle\, \delta(u_{B_i}^{-1}u_{A_i}u_{H_{i-1}}\cdots u_{M_1})(u_{B_i}^{-1}u_{A_i}u_{H_{i-1}}\cdots u_{M_1})^{-1}\,,\, \boldsymbol{j}_{B_i}\rangle,
\end{aligned} \tag{5.6}$$

where $\quad u_{H_i} = [u_{B_i}, u_{A_i}^{-1}] = u_{B_i}u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}.$

Proof:

The proof is a rather lengthy calculation. We outline the main steps and some auxiliary identities used in the calculations.

1. To evaluate the sum $\sum_{i=1}^{n+2g}\langle\delta K_i K_i^{-1}\wedge\delta K_{i-1}K_{i-1}^{-1}\rangle$, one recalls that the group elements K_i are given by

$$\begin{aligned}
&K_i = M_1^{-1}\cdots M_i^{-1} \qquad 1\leq i\leq n\\
&K_{n+2i-1} = M_1^{-1}\cdots M_n^{-1}[B_1,A_1^{-1}]^{-1}\cdots[B_{i-1},A_{i-1}^{-1}]^{-1}A_i^{-1}\\
&K_{n+2i} = M_1^{-1}\cdots M_n^{-1}[B_1,A_1^{-1}]^{-1}\cdots[B_i,A_i^{-1}]^{-1} \qquad\qquad 1\leq i\leq g.
\end{aligned}\tag{5.7}$$

Inserting the parametrisation (5.3) of the holonomies along the generators of the fundamental group, one can express these group elements as functions of the variables $u_{M_i},\boldsymbol{j}_{M_i}$, $u_{A_j},\boldsymbol{j}_{A_j}$ and $u_{B_j},\boldsymbol{j}_{B_j}$ in (5.3)

$$K_i^{-1} =: (u_{K_i}, -\mathrm{Ad}^*(u_{K_i}^{-1})\boldsymbol{j}_{K_i}) \qquad 1\leq i\leq n+2g \tag{5.8}$$

with

$$u_{K_i} = \begin{cases} u_{M_i}\cdots u_{M_1} & 1\leq i\leq n\\ u_{A_j}u_{H_{j-1}}\cdots u_{H_1}u_{M_n}\cdots u_{M_1} & i=n+2j-1\\ u_{H_j}\cdots u_{H_1}u_{M_n}\cdots u_{M_1} & i=n+2j\end{cases}\tag{5.9}$$

$$\begin{aligned}
&\boldsymbol{j}_{K_i} = \sum_{j=1}^{i}\mathrm{Ad}^*(u_{M_{j-1}}...u_{M_1})\boldsymbol{j}_{M_j} \qquad\qquad 1\leq i\leq n\\
&\boldsymbol{j}_{K_{n+2i-1}} = \mathrm{Ad}^*(u_{H_{i-1}}...u_{M_1})\boldsymbol{j}_{A_i} + \sum_{k=1}^{n}\mathrm{Ad}^*(u_{M_{k-1}}...u_{M_1})\boldsymbol{j}_{M_k} + \sum_{k=1}^{i-1}\mathrm{Ad}^*(u_{H_{k-1}}...u_{M_1})\boldsymbol{j}_{H_k}\\
&\boldsymbol{j}_{K_{n+2i}} = \mathrm{Ad}^*(u_{H_{i-1}}...u_{M_1})\boldsymbol{j}_{H_i} + \sum_{k=1}^{n}\mathrm{Ad}^*(u_{M_{k-1}}...u_{M_1})\boldsymbol{j}_{M_k} + \sum_{k=1}^{i-1}\mathrm{Ad}^*(u_{H_{k-1}}...u_{M_1})\boldsymbol{j}_{H_k},
\end{aligned}\tag{5.10}$$

$$\begin{aligned}
&u_{H_i} := u_{B_i}u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}\\
&\boldsymbol{j}^{H_i} := (1-\mathrm{Ad}^*(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}))\boldsymbol{j}^{A_i} + (\mathrm{Ad}^*(u_{A_i}^{-1}u_{B_i}^{-1}u_{A_i}) - \mathrm{Ad}^*(u_{B_i}^{-1}u_{A_i}))\boldsymbol{j}^{B_i}.
\end{aligned}\tag{5.11}$$

One then computes the derivatives $\delta K_i K_i^{-1}$ using the identities

$$\begin{aligned}
&(v,\boldsymbol{x})^{-1}\delta(v,\boldsymbol{x}) = (v^{-1}\delta v, \mathrm{Ad}^*(v)\delta\boldsymbol{x}) \qquad \forall v\in G, \boldsymbol{x}\in\mathfrak{g}^*\\
&\delta(\mathrm{Ad}^*(v^{-1})z) = \mathrm{Ad}^*(v^{-1})\delta z + \mathrm{Ad}^*(v^{-1})[v^{-1}\delta v, z] \qquad \forall v\in G, z\in\mathfrak{g}^*.
\end{aligned}\tag{5.12}$$

Taking into account that the bilinear form $\langle\,,\,\rangle$ pairs the $\mathfrak{g}$-component of the first argument with the $\mathfrak{g}^*$-component of the second and vice versa, one obtains

$$\begin{aligned}
&\sum_{i=1}^{n+2g}\langle\delta K_iK_i^{-1}\wedge\delta K_{i-1}K_{i-1}^{-1}\rangle=\langle\delta\boldsymbol{j}_{K_{n+2g}},u_{K_{n+2g}}^{-1}\delta u_{K_{n+2g}}\rangle-\sum_{i=1}^{n}\langle\delta\boldsymbol{j}_{M_i},u_{M_i}^{-1}\delta u_{M_i}\rangle \\
&-\sum_{i=1}^{g}\langle\delta\boldsymbol{j}_{A_i},u_{A_i}^{-1}\delta u_{A_i}\rangle+\langle\delta\left(\mathrm{Ad}^*(u_{A_i}^{-1}u_{B_i}^{-1})(\boldsymbol{j}_{B_i}-\boldsymbol{j}_{A_i})-\mathrm{Ad}^*(u_{B_i}^{-1})\boldsymbol{j}_{B_i}\right),u_{B_i}u_{A_i}u_{B_i}^{-1}\delta(u_{B_i}u_{A_i}^{-1}u_{B_i}^{-1})\rangle \\
&-2\sum_{i=1}^{n}\delta\langle\boldsymbol{j}_{M_i},\delta(u_{M_{i-1}}\cdots u_{M_1})u_{M_1}^{-1}\cdots u_{M_{i-1}}^{-1}\rangle-2\sum_{i=1}^{2g}\delta\langle\boldsymbol{j}_{K_{n+i}},\delta u_{K_{n+i-1}}u_{K_{n+i-1}}^{-1}\rangle.
\end{aligned}\tag{5.13}$$

2. For the terms containing elements C_i, $i=1,\ldots,n$ one uses identities (4.27) which express the holonomies M_i around the punctures in terms of the variables g_i and C_i together with Lemma 5.1. Taking into account that $\delta C_i=0$ for $i=1,\ldots,n$ and we find that the terms of the form $\langle\delta\boldsymbol{j},u^{-1}\delta u\rangle$ in (5.13) and (5.2) cancel. To evaluate the terms involving A_j,B_j, $j=1,\ldots,g$, one inserts the parametrisation (5.3) of the holonomies A_j,B_j into the second line of (4.36) and finds

$$\begin{aligned}
&-\langle A_i^{-1}\delta A_i\wedge B_i^{-1}\delta B_i\rangle-\langle\delta(B_iA_iB_i^{-1})B_iA_i^{-1}B_i^{-1}\wedge\delta B_i\,B_i^{-1}\rangle= \\
&\qquad\langle\delta\left(\mathrm{Ad}^*(u_{B_i}^{-1})(\boldsymbol{j}_{B_i}-\boldsymbol{j}_{A_i})-\mathrm{Ad}^*(u_{A_i}u_{B_i}^{-1})\boldsymbol{j}_{B_i}\right)\,,\,\delta(u_{B_i}u_{A_i}^{-1}u_{B_i}^{-1})u_{B_i}u_{A_i}u_{B_i}^{-1}\rangle \\
&\qquad-2\delta\langle\boldsymbol{j}_{B_i}-\boldsymbol{j}_{A_i}\,,\,u_{B_i}^{-1}\delta u_{B_i}\rangle+2\delta\langle\boldsymbol{j}_{B_i}\,,\,u_{A_i}u_{B_i}^{-1}\delta(u_{B_i}u_{A_i}^{-1})\rangle-\langle\delta\boldsymbol{j}_{A_i}\,,\,u_{A_i}^{-1}\delta u_{A_i}\rangle.
\end{aligned}\tag{5.14}$$

Adding this term to the second line in (5.13) and then using the definition (5.10) to evaluate the last term in (5.13) then yields

$$\begin{aligned}
\Omega=&\tfrac{k}{2\pi}\delta\langle(\boldsymbol{\mu},\boldsymbol{s}),w^{-1}\delta w\rangle+\tfrac{k}{2\pi}\delta\Theta-\tfrac{k}{4\pi}\langle\delta\boldsymbol{j}_{K_{n+2g}}\,,\,u_{K_{n+2g}}^{-1}\delta u_{K_{n+2g}}\rangle \\
&+\tfrac{k}{4\pi}\langle K_{n+2g}^{-1}\delta K_{n+2g},\delta g_\infty g_\infty^{-1}\rangle+\tfrac{k}{4\pi}\langle(\delta\boldsymbol{\mu},\delta\boldsymbol{s})\,,\,g_\infty^{-1}\delta g_\infty\rangle.
\end{aligned}\tag{5.15}$$

3. To simplify the remaining terms, we use identity (4.28) for the holonomy around the distinguished puncture and, again, Lemma 5.1, this time with $K=K_{n+2g}$, $g=g_\infty=(v_\infty,\boldsymbol{x}_\infty)$, $C=\exp(\frac{2\pi}{k}D)=(e^{-\boldsymbol{\mu}},-\boldsymbol{s})$. This allows one to eliminate the last three terms in (5.15) and finally gives

$$\begin{aligned}
\Omega=&\tfrac{k}{2\pi}\,\delta\langle(\boldsymbol{\mu},\boldsymbol{s}),w^{-1}\delta w\rangle+\tfrac{k}{2\pi}\,\delta\Theta+\tfrac{k}{4\pi}\langle\delta\boldsymbol{s},\delta\boldsymbol{\mu}\rangle+\tfrac{k}{2\pi}\,\delta\langle\boldsymbol{j}_{K_{n+2g}},\delta v_\infty v_\infty^{-1}\rangle \\
&-\tfrac{k}{2\pi}\,\delta\langle\mathrm{Ad}^*(v_\infty)\boldsymbol{x}_\infty,\delta\boldsymbol{\mu}\rangle. \qquad\Box
\end{aligned}\tag{5.16}$$

The expression (5.16), or equivalently (5.5), is an explicit formula for the pull-back of the symplectic form on the moduli space to the extended phase space. The pull-back is the exterior derivative of a sum of one-forms which deserve further comment. As explained in Appendix A, the first term in (5.16) is a reduction of the symplectic structure on the contangent bundle of $G\ltimes\mathfrak{g}^*$ while the last two terms in the expression (5.16) are a reduction of the canonical symplectic structure on the Heisenberg double of $G\ltimes\mathfrak{g}^*$, see the discussion

preceding (A.24). The symplectic potential Θ contains the contributions from the handles and the ordinary punctures. It was first given in [7] in the context of an investigation of Poisson structures arising in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$ on $\mathbb{R} \times S_{g,n}$, where $S_{g,n}$ is a surface of genus g and with n ordinary punctures. Applying results from [8] it was shown there that in suitable "decoupled" coordinates Θ is a sum of standard symplectic potentials, namely a copy of the symplectic potential of the Heisenberg double (A.24) for every handle, and a copy of the pull-back of the symplectic potential on the conjugacy classes of $G \ltimes \mathfrak{g}^*$. As explained after (A.24) the latter are the symplectic leaves of the dual Poisson-Lie group $G \times \mathfrak{g}^*$.

The structural relationship between the two-form (5.16) and the Heisenberg double (as well as the various symplectic forms derived from it) generalises to arbitrary Poisson-Lie groups. However, the explicit and rather simple formula we were able to give for the two-form Ω is a reflection of the relatively simple structure of Poisson-Lie groups of the form $G \ltimes \mathfrak{g}^*$. We will point out some uses of the formula (5.16) the Chern-Simons formulation of (2+1)-dimensional gravity in our conclusion.

6 Gauge invariance, basepoint independence and gauge fixing

Theorem 5.2 gives an expression for the two-form (5.5) in terms of the $G \ltimes \mathfrak{g}^*$ elements parametrising the holonomies with respect to given basepoint p_0 and the variables $w \in G \ltimes \mathfrak{g}^*$, $(e^{-\boldsymbol{\mu}}, -\boldsymbol{s}) \in T_{\mathfrak{c}}$. As explained at the beginning of Sect. 4, this two-form is the pull-back of the symplectic form on the moduli space but not itself symplectic. This is due to the fact that the parametrisation of the phase space in terms of the variables in (5.5) exhibits a two-fold redundancy.

First, the holonomy variables M_i, A_j, B_j, L_∞ are redundant, as they are subject to the constraint (4.25). The associated gauge transformations act on the holonomies by simultaneously conjugating them with a general $G \ltimes \mathfrak{g}^*$-valued function of M_i, A_j, B_j, L_∞. Such gauge transformations arise from Chern-Simons gauge transformations of the form (3.10) that are nontrivial at the basepoint p_0. We will see below that they also describe the transformation of the holonomies under a change of basepoint. In this section, we will demonstrate the gauge invariance of the two-form (5.5) under conjugation of all holonomies. It then follows in particular that the symplectic structure of the moduli space is independent of the base point. When the gauge is partly fixed by requiring the curvature at the distinguished puncture to lie in the Cartan subalgebra $\mathfrak{c}$, the two-form (5.5) takes a particularly simple form.

The second redundancy is linked to the fact that the parametrisation of the holonomies M_i and L_∞ in terms of general group elements $g_i, g_\infty \in G \ltimes \mathfrak{g}^*$, $C \in T_{\mathfrak{c}}$ and (constant) elements $C_i \in \cup_{\iota \in I} T_{\mathfrak{c}_\iota}$, see (4.27), (4.28) is not unique. Right-multiplication of the elements g_i and g_∞ in (4.27), (4.28) by elements of the stabiliser group of C_i and C yields the same expression for the holonomies M_i, L_∞. The group elements $g_i, g_\infty \in G \ltimes \mathfrak{g}^*$ are therefore only defined up to right-multiplication with elements of the relevant stabilisers, which gives rise to additional gauge transformations discussed at the end of this section.

Before we enter the detailed calculations, we note that gauge invariance manifests itself in the degeneracy of Ω restricted to the constraint surface $\mathcal{C}$, i. e. in the existence of a vector field Z on $\mathcal{C}$ such that the contraction of Ω with Z is zero. The vector field Z is the infinitesimal generator of the gauge transformation. In our case Ω is exact, so $\Omega = \delta\chi$ and Z is an infinitesimal gauge transformation if and only if

$$\iota_Z \delta\chi = 0 \Leftrightarrow \mathcal{L}_Z \chi - \delta\iota_Z\chi = 0, \tag{6.1}$$

where $\mathcal{L}_Z$ is the Lie derivative with respect to Z, and we have used Cartan's formula. It turns out that all the gauge transformations we encounter in this section are group actions ρ of $G \ltimes \mathfrak{g}^*$ or a subgroup of $G \ltimes \mathfrak{g}^*$ on $\mathcal{C}$. For calculations we have found it convenient to consider finite rather than infinitesimal group actions. If $h(\epsilon)$ is a one-parameter subgroup of $G \ltimes \mathfrak{g}^*$ with $h(0) = 1$ and Z the vector field generated by the action of $\rho(h(\epsilon))$ on $\mathcal{C}$, then $\mathcal{L}_Z\chi = 0$ if χ is invariant under pull-back with $\rho(h(\epsilon))$; this turns out to be the case for most gauge transformations considered here. The gauge invariance condition (6.1) then requires that the function $\iota_Z\chi = \chi(Z)$ is constant. More generally, if $\iota_Z\chi = H$ is a non-constant function on $\mathcal{C}$, the condition (6.1) is satisfied if $\mathcal{L}_Z\chi = \delta H$.

The quickest way of checking if (6.1) holds is to allow the parameter ϵ in the action $\rho(h(\epsilon))$ to be a function on the constraint surface, and to compute the changes in χ under the pull-back with $\rho(h(\epsilon))$. If χ changes according to

$$\chi \mapsto \chi + \delta(\epsilon H) + \mathcal{O}(\epsilon^2) \tag{6.2}$$

for some function H, it follows that $H = \chi(Z)$ and (by specialising to a $\mathcal{C}$-independent parameter ϵ) that $\mathcal{L}_Z\chi = \delta H$. Hence (6.2) implies (6.1). In the following we do not specify the one-parameter subgroup $h(\epsilon)$ but instead consider group actions of general group elements $g \in G \ltimes \mathfrak{g}^*$ which are functions on $\mathcal{C}$. From the transformation behaviour of χ under this action we can then read off the behaviour under any one-parameter subgroups with a parameter ϵ which is a function of $\mathcal{C}$.

We start by considering the transformation of the two-form (5.5) under conjugation of all holonomies by $g \in G \ltimes \mathfrak{g}^*$ which depends on the extended phase space variables. The transformation behaviour of the various terms in (5.5) can be stated in two technical lemmas which can be proved using the following general formula for conjugation in $G \ltimes \mathfrak{g}^*$. Suppose $M = (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j})$ and $\tilde{M} = (\tilde{u}, -\mathrm{Ad}^*(\tilde{u}^{-1})\tilde{\mathbf{j}})$ are related by $M = g\tilde{M}g^{-1}$, with $g = (v, \boldsymbol{x})$. Then

$$\boldsymbol{j} = (1 - \mathrm{Ad}^*(v\tilde{u}v^{-1}))\boldsymbol{x} + \mathrm{Ad}^*(v^{-1})\tilde{\mathbf{j}}, \quad u = v\tilde{u}v^{-1}. \tag{6.3}$$

The first lemma concerns the last two terms in (5.16). We state it in a slightly more general form.

Lemma 6.1 *Suppose* $(\boldsymbol{\mu}, \boldsymbol{s})$ *is an arbitrary element in the Cartan subalgebra* $\mathfrak{c}$ *of* $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ *(not fixed), and the* $G \ltimes \mathfrak{g}^*$ *elements* $g_K = (v_K, \boldsymbol{x}_K)$ *and* $\tilde{g}_K = (\tilde{v}_K, \tilde{\boldsymbol{x}}_K)$ *are related by*

left-multiplication $g_K = g\tilde{g}_K$*, where* $g = (v, \boldsymbol{x})$ *is a function of* $\tilde{g}_K$ *and* $(\boldsymbol{\mu}, \boldsymbol{s})$*. Defining*

$$\begin{aligned} K^{-1} &= (u_K, -\mathrm{Ad}^*(u_K^{-1})\boldsymbol{j}_K) = g_K(e^{-\boldsymbol{\mu}}, -\boldsymbol{s})g_K^{-1} \\ \tilde{K}^{-1} &= (\tilde{u}_K, -\mathrm{Ad}^*(\tilde{u}_K^{-1})\tilde{\mathbf{j}}_K) = \tilde{g}_K(e^{-\boldsymbol{\mu}}, -\boldsymbol{s})\tilde{g}_K^{-1} \end{aligned} \tag{6.4}$$

so that $K = g\tilde{K}g^{-1}$*, we have the identities*

$$\begin{aligned} & \langle \boldsymbol{j}_K, \delta v_K v_K^{-1}\rangle - \langle \mathrm{Ad}^*(v_K)\boldsymbol{x}_K, \delta\boldsymbol{\mu}\rangle \\ = \; & \langle \tilde{\mathbf{j}}_K, \delta\tilde{v}_K\tilde{v}_K^{-1}\rangle - \langle \mathrm{Ad}^*(\tilde{v}_K)\tilde{\boldsymbol{x}}_K, \delta\boldsymbol{\mu}\rangle + \langle \boldsymbol{j}_K, \delta v v^{-1}\rangle + \langle \mathrm{Ad}^*(v)\boldsymbol{x}, \delta\tilde{u}_K\tilde{u}_K^{-1}\rangle, \\ = \; & \langle \tilde{\mathbf{j}}_K, \delta\tilde{v}_K\tilde{v}_K^{-1}\rangle + \langle \mathrm{Ad}^*(\tilde{v}_K)\tilde{\boldsymbol{x}}_K, \delta\boldsymbol{\mu}\rangle + \langle \tilde{\mathbf{j}}_K, v^{-1}\delta v\rangle + \langle \boldsymbol{x}\,,\, \delta u_K u_K^{-1}\rangle. \end{aligned} \tag{6.5}$$

The lemma can be proved by straightforward calculation. Comparison with (A.23) shows that the last two terms in (6.5) are precisely the symplectic form on the Heisenberg double of $G \ltimes \mathfrak{g}^*$. In the terminology of Appendix A.2 this lemma states that the symplectic form (A.24) (itself a reduction of the Heisenberg double symplectic form) on the $G \ltimes \mathfrak{g}^*$ element M changes by the Heisenberg double symplectic form for g and M when M is conjugated by a function g.

The second lemma concerns the symplectic potential Θ in (5.5) and shows that it has an equal but opposite transformation behaviour under conjugation of all the holonomies.

Lemma 6.2 *If the group elements* M_i*,* A_j*,* $B_j \in G \ltimes \mathfrak{g}^*$ *are obtained from* $\tilde{M}_i, \tilde{A}_j, \tilde{B}_j$ *via simultaneous conjugation with a* $G \ltimes \mathfrak{g}^*$*-valued function* $g = (v, \boldsymbol{x})$ *on the extended phase space*

$$\begin{aligned} M_i &= (u_{M_i}, -\mathrm{Ad}^*(u_{M_i}^{-1})\boldsymbol{j}_{M_i}) = g\tilde{M}_i g^{-1} = g \cdot (\tilde{u}_{M_i}, -\mathrm{Ad}^*(\tilde{u}_{M_i}^{-1})\tilde{\mathbf{j}}_{M_i}) \cdot g^{-1} \\ A_j &= (u_{A_j}, -\mathrm{Ad}^*(u_{A_j}^{-1})\boldsymbol{j}_{A_j}) = g\tilde{A}_j g^{-1} = g \cdot (\tilde{u}_{A_j}, -\mathrm{Ad}^*(\tilde{u}_{A_j}^{-1})\tilde{\mathbf{j}}_{A_j}) \cdot g^{-1} \\ B_j &= (u_{B_j}, -\mathrm{Ad}^*(u_{B_j}^{-1})\boldsymbol{j}_{B_j}) = g\tilde{B}_j g^{-1} = g \cdot (\tilde{u}_{B_j}, -\mathrm{Ad}^*(\tilde{u}_{B_j}^{-1})\tilde{\mathbf{j}}_{B_j}) \cdot g^{-1}, \end{aligned} \tag{6.6}$$

and the variables v_{M_i} *defined in* (5.3) *from* $\tilde{v}_{M_i}$ *via left-multiplication with the Lorentz component of this function*

$$v_{M_i} = v\tilde{v}_{M_i} \tag{6.7}$$

then the expressions for the symplectic potential Θ *in terms of these two sets of variables are related by*

$$\begin{aligned} & \Theta(v_{M_i}, \boldsymbol{j}_{M_i}, u_{A_j}, \boldsymbol{j}_{A_j}, u_{B_j}, \boldsymbol{j}_{B_j}) \\ & = \Theta(\tilde{v}_{M_i}, \tilde{\mathbf{j}}_{M_i}, \tilde{u}_{A_j}, \tilde{\mathbf{j}}_{A_j}, \tilde{u}_{B_j}, \tilde{\mathbf{j}}_{B_j}) - \langle \boldsymbol{j}_{K_{n+2g}}\,,\, \delta v v^{-1}\rangle - \langle \mathrm{Ad}^*(v)\boldsymbol{x}\,,\, \delta\tilde{u}_{K_{n+2g}}\tilde{u}_{K_{n+2g}}^{-1}\rangle \\ & = \Theta(\tilde{v}_{M_i}, \tilde{\mathbf{j}}_{M_i}, \tilde{u}_{A_j}, \tilde{\mathbf{j}}_{A_j}, \tilde{u}_{B_j}, \tilde{\mathbf{j}}_{B_j}) - \langle \tilde{\mathbf{j}}_{K_{n+2g}}\,,\, v^{-1}\delta v\rangle - \langle \boldsymbol{x}\,,\, \delta u_{K_{n+2g}} u_{K_{n+2g}}^{-1}\rangle, \end{aligned} \tag{6.8}$$

where $u_{K_{n+2g}}$*,* $\boldsymbol{j}_{K_{n+2g}}$*,* $\tilde{u}_{K_{n+2g}}$*,* $\tilde{\mathbf{j}}_{K_{n+2g}}$ *are given in terms of the variables* M_i, A_j, B_j *and* $\tilde{M}_i, \tilde{A}_j, \tilde{B}_j$*, respectively, by* (5.4)*,* (5.9) *and* (5.10)*.*

Applying Lemma 6.1 to $K = K_{n+2g}$ and comparing (6.5) and (6.8), we see that the terms arising from the transformation are equal, but with opposite sign. With the considerations above we therefore find that such transformations are gauge symmetries of the two-form Ω. Furthermore, by applying Lemma 6.2 with $g = g_\infty$, it is possible to transform the holonomy variables into a new set of variables in which the two-form Ω takes a particularly simple form. We obtain the following theorem.

Theorem 6.3

1. Simultaneous conjugation of the holonomies M_i, $i = 1, \ldots, n$, A_j, B_j, $j = 1, \ldots, g$ and K_{n+2g}, and the left-multiplication of $(v_{M_i}, \boldsymbol{x}_{M_i})$, and $g_\infty = (v_\infty, \boldsymbol{x}_\infty)$ by an arbitrary element $g \in G \ltimes \mathfrak{g}^*$ is a gauge transformation for the two-form (5.5)*.*

2. In terms of the variables $\tilde{v}_{M_i}, \tilde{\mathbf{j}}_{M_i}, \tilde{u}_{A_j}, \tilde{\mathbf{j}}_{A_j}, \tilde{u}_{B_j}, \tilde{\mathbf{j}}_{B_j}$ defined by

$$\begin{aligned} \tilde{M}_i &= (\tilde{u}_{M_i}, -\mathrm{Ad}^*(\tilde{u}_{M_i}^{-1})\tilde{\mathbf{j}}_{M_i}) = g_\infty^{-1} M_i g_\infty \\ \tilde{A}_i &= (\tilde{u}_{A_i}, -\mathrm{Ad}^*(\tilde{u}_{A_i}^{-1})\tilde{\mathbf{j}}_{A_i}) = g_\infty^{-1} A_i g_\infty \\ \tilde{B}_i &= (\tilde{u}_{B_i}, -\mathrm{Ad}^*(\tilde{u}_{B_i}^{-1})\tilde{\mathbf{j}}_{B_i}) = g_\infty^{-1} B_i g_\infty \\ \tilde{v}_{M_i} &= v_\infty^{-1} v_{M_i} \end{aligned} \tag{6.9}$$

the two-form (5.5) *is*

$$\Omega = \tfrac{k}{2\pi}\delta\left(\langle(\boldsymbol{\mu}, \boldsymbol{s}), w^{-1}\delta w\rangle - \tfrac{1}{2}\langle\boldsymbol{\mu}\delta\boldsymbol{s}\rangle + \Theta(\tilde{v}_{M_i}, \tilde{\mathbf{j}}_{M_i}, \tilde{u}_{A_j}, \tilde{\mathbf{j}}_{A_j}, \tilde{u}_{B_j}, \tilde{\mathbf{j}}_{B_j})\right). \tag{6.10}$$

In particular, this theorem addresses the dependence of the two-form Ω on the choice of the basepoint. Under a change of basepoint $p_0 \mapsto p_0'$, the holonomy of any closed curve $\gamma \in \pi_1(S_{g,n}^\infty, p_0)$ with respect to basepoint p_0 transforms according to

$$H[\gamma, p_0] \mapsto H[\gamma, p_0'] = PT_{p_0 \to p_0'} \cdot H[\gamma, p_0] \cdot PT_{p_0' \to p_0}, \tag{6.11}$$

where $PT_{x \to y}$ denotes the $G \ltimes \mathfrak{g}^*$-element which implements the parallel transport from x to y. It follows from the first part of Theorem (6.3) that expressions (5.5) for the two-form Ω derived with respect to two different basepoints agree, and we have

Corollary 6.4 *The two-form* (5.5) *does not depend on the choice of basepoint p_0 of the fundamental group $\pi_1(S_{g,n}, p_0)$.*

Note that the holonomies defined in the second part of Theorem 6.3 satisfy

$$L_\infty^{-1} = [\tilde{B}_g, \tilde{A}_g^{-1}] \cdots [\tilde{B}_1, \tilde{A}_1^{-1}] \cdot \tilde{M}_n \cdots \tilde{M}_1 = e^{\frac{2\pi}{k}D} = (e^{-\boldsymbol{\mu}}, -\boldsymbol{s}). \tag{6.12}$$

They are therefore the holonomies in a gauge where the holonomy around the distinguished puncture is in the abelian subgroup $T_{\mathfrak{c}}$. This is the partial gauge fixing anticipated in the opening paragraph of this subsection. However, as discussed above, there is a residual gauge freedom linked to the redundancy of the parametrisation of the holonomies M_i, L_∞ in terms

of general group elements and elements of the subgroups $T_{\mathfrak{c}_\iota}$. For the holonomy L_∞, the condition (6.12) does not eliminate the gauge invariance of (5.5) completely. The condition is preserved by transformations of the form

$$\begin{aligned} &w \mapsto wg \\ &\tilde{X} \mapsto g\tilde{X}g^{-1} \qquad \forall \tilde{X} \in \{\tilde{M}_1, \dots, \tilde{B}_g\} \end{aligned} \tag{6.13}$$

for any element $g = (v, \boldsymbol{x})$ in the stabiliser group of $(\boldsymbol{\mu}, \boldsymbol{s})$, i.e. $\mathrm{Ad}(v)\boldsymbol{\mu} = \boldsymbol{\mu}$ and $\mathrm{Ad}^*(v^{-1})\boldsymbol{s} = \boldsymbol{s}$. To see that these transformation are gauge transformations of (6.10) note the following transformation properties of the symplectic potentials entering (6.10):

$$\langle(\boldsymbol{\mu}, \boldsymbol{s}), w^{-1}\delta w\rangle \mapsto \langle(\boldsymbol{\mu}, \boldsymbol{s}), w^{-1}\delta w\rangle + \langle\boldsymbol{\mu}, \mathrm{Ad}^*(v)\delta\boldsymbol{x}\rangle + \langle\boldsymbol{s}, v^{-1}\delta v\rangle. \tag{6.14}$$

From Lemma (6.2) we have

$$\begin{aligned} &\Theta(\tilde{v}_{M_i}, \tilde{\mathbf{j}}_{M_i}, \tilde{u}_{A_j}, \tilde{\mathbf{j}}_{A_j}, \tilde{u}_{B_j}, \tilde{\mathbf{j}}_{B_j}) \mapsto \\ &\qquad \Theta(\tilde{v}_{M_i}, \tilde{\mathbf{j}}_{M_i}, \tilde{u}_{A_j}, \tilde{\mathbf{j}}_{A_j}, \tilde{u}_{B_j}, \tilde{\mathbf{j}}_{B_j}) - \langle\boldsymbol{s}, \delta v v^{-1}\rangle + \langle\mathrm{Ad}^*(v)\boldsymbol{x}, \delta\boldsymbol{\mu}\rangle, \end{aligned} \tag{6.15}$$

and, using the stabiliser property of g, we check that

$$\langle\mathrm{Ad}^*(v)\boldsymbol{x}, \delta\boldsymbol{\mu}\rangle = \delta\langle\mathrm{Ad}^*(v)\boldsymbol{x}, \boldsymbol{\mu}\rangle - \langle\boldsymbol{\mu}, \mathrm{Ad}^*(v)\delta\boldsymbol{x}\rangle + \langle\boldsymbol{s}, \delta v v^{-1}\rangle - \langle\boldsymbol{s}, v^{-1}\delta v\rangle, \tag{6.16}$$

so that the sum $\Theta + \langle(\boldsymbol{\mu}, \boldsymbol{s}), w^{-1}\delta w\rangle$ changes according to

$$\Theta + \langle(\boldsymbol{\mu}, \boldsymbol{s}), w^{-1}\delta w\rangle \mapsto \Theta + \langle(\boldsymbol{\mu}, \boldsymbol{s}), w^{-1}\delta w\rangle + \delta\langle\mathrm{Ad}^*(v)\boldsymbol{x}, \boldsymbol{\mu}\rangle. \tag{6.17}$$

For each one-parameter subgroup $h(\epsilon)$ of the stabiliser group of $(\boldsymbol{\mu}, \boldsymbol{s})$ this transformation behaviour is of the form (6.2), showing that the transformations (6.13) are indeed gauge transformations.

Similar residual gauge transformations arise for each of the ordinary punctures. The redundancy (3.11) in the parametrisation (3.2) of coadjoint orbits leads to the gauge transformation

$$\tilde{v}_{M_i} \mapsto \tilde{v}_{M_i} v \tag{6.18}$$

where v is such that $(v, 0)$ is in the stabiliser group of $(\boldsymbol{\mu}_i, \boldsymbol{s}_i)$, i.e. $\mathrm{Ad}(v)\boldsymbol{\mu}_i = \boldsymbol{\mu}_i$ and $\mathrm{Ad}^*(v^{-1})\boldsymbol{s}_i = \boldsymbol{s}_i$. Under the transformation (6.18) the symplectic potential (5.6) changes according to

$$\Theta(\tilde{v}_{M_i}, \tilde{\mathbf{j}}_{M_i}, \tilde{u}_{A_j}, \tilde{\mathbf{j}}_{A_j}, \tilde{u}_{B_j}, \tilde{\mathbf{j}}_{B_j}) \mapsto \Theta(\tilde{v}_{M_i}, \tilde{\mathbf{j}}_{M_i}, \tilde{u}_{A_j}, \tilde{\mathbf{j}}_{A_j}, \tilde{u}_{B_j}, \tilde{\mathbf{j}}_{B_j}) - \langle\boldsymbol{s}_i, \delta v v^{-1}\rangle \tag{6.19}$$

Again we see that for each one-parameter subgroup $h(\epsilon)$ of the stabiliser group this transformation behaviour is of the form (6.2), showing that the transformations (6.18) are indeed gauge transformations.

The physical phase space $\mathcal{P}_{phys}$ is, by definition, the quotient of the constraint surface $\mathcal{C}$ by the gauge transformations discussed in this section. It is difficult to analyse $\mathcal{P}_{phys}$ in more

detail without restricting the group $G \ltimes \mathfrak{g}^*$ to a smaller class or particular example. Even for compact gauge groups it is known that the physical phase space of Chern-Simons theory is a manifold with singularities, see [17]. Because of the non-compactness of $G \ltimes \mathfrak{g}^*$ it may happen that $\mathcal{P}_{phys}$ is not even a Hausdorff space, see [18] for a discussion of an example. While we study the case of the Poincaré group in (2+1) dimensions in more detail in [6], here we only count the dimension of $\mathcal{P}_{phys}$, assuming that it is a manifold. The variables obtained after partial gauge fixing are the $G \ltimes \mathfrak{g}^*$-elements w, $\tilde{M}_i$ $i = 1, \dots, n$, $\tilde{A}_j, \tilde{B}_j$, $j = 1, \dots, g$ and the $\mathfrak{c}$-element $(\boldsymbol{\mu}, \boldsymbol{s})$. The holonomies $\tilde{M}_i$ are each restricted to lie in conjugacy classes

$$\mathcal{C}_{\boldsymbol{\mu}_i, \boldsymbol{s}_i} = \{g(e^{-\boldsymbol{\mu}_i}, -\boldsymbol{s}_i)g^{-1} | g \in G \ltimes \mathfrak{g}^*\}, \tag{6.20}$$

and we have to impose the constraint (6.12) and divide by the gauge transformation (6.13). The variables $(\boldsymbol{\mu}, \boldsymbol{s})$ are good coordinates on the physical phase space in the generic situation where the stabiliser group of $(\boldsymbol{\mu}, \boldsymbol{s})$ has the same dimension as $\mathfrak{c}$. Thus we obtain the dimension of the physical phase space as follows:

$$\begin{aligned} \dim \mathcal{P}_{phys} &= \sum_{i=1}^{n} \dim(\mathcal{C}_{\boldsymbol{\mu}_i, \boldsymbol{s}_i}) + (2g+1) \cdot \dim(G \ltimes \mathfrak{g}^*) + \dim \mathfrak{c} - (\dim(G \ltimes \mathfrak{g}^*) + \dim \mathfrak{c}) \\ &= \sum_{i=1}^{n} \dim(\mathcal{C}_{\boldsymbol{\mu}_i, \boldsymbol{s}_i}) + 2g \cdot \dim(G \ltimes \mathfrak{g}^*). \end{aligned} \tag{6.21}$$

7 Outlook and conclusion

In this paper we introduced a new way of treating a puncture in Chern-Simons theory and explicitly determined the symplectic structure on the phase space of Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$ on $\mathbb{R} \times S^\infty_{g,n}$ when one distinguished puncture on $S^\infty_{g,n}$ is treated in this way. Our main motivation for studying gauge groups of the form $G \ltimes \mathfrak{g}^*$ is the application to (2+1)-dimensional gravity, which can be formulated as a Chern-Simons theory with gauge groups of the form $G \ltimes \mathfrak{g}^*$ when the cosmological constant is zero.

The application to (2+1)-dimensional gravity is described in detail in the paper [6], where we use a distinguished puncture of the type considered here to model open universes. In this context, the ordinary punctures represent particles of mass $\boldsymbol{\mu}_i$ and spin $\boldsymbol{s}_i$ in a spacetime of genus g. The distinguished puncture stands for the boundary of the universe at spatial infinity, and the phase space variables $\boldsymbol{\mu}$ and $\boldsymbol{s}$ are interpreted as total mass and spin of the universe. The two-form (5.5) and the explicit parametrisation of the extended phase therefore provide the basis for a detailed study of the classical dynamics of an open universe containing an arbitrary number of particles. Moreover, it can serve as the starting point for an equally explicit study of (2+1)-dimensional quantum gravity.

Although many aspects of the model considered here were motivated by the application to (2+1)-dimensional gravity, our treatment of the distinguished puncture makes sense for any gauge group satisfying the technical assumptions spelled out in Sect. 2. In a more general context, it may also be of interest to consider several distinguished punctures. It is not

difficult to generalise our discussion in Sect. 4 to take into account several distinguished punctures and the additional boundary components which would result from the cutting procedure described there.

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A Poisson structures associated to $G \ltimes \mathfrak{g}^*$

This appendix summarises properties of two symplectic structures which play an important role in this paper. The first is the canonical symplectic structure defined on the cotangent bundle of groups of the form $G \ltimes \mathfrak{g}^*$, and the second is the symplectic structure on the Heisenberg double of $G \ltimes \mathfrak{g}^*$. While the cotangent bundle symplectic structure is defined for any Lie group, the Heisenberg double structure is linked to the properties of $G \ltimes \mathfrak{g}^*$ as a Poisson-Lie group, see [14, 15] and the textbook [16]. We show how various symplectic structures which arise in the paper can be deduced from the cotangent bundle and Heisenberg double symplectic structures.

A.1 Cotangent bundle of $G \ltimes \mathfrak{g}^*$

In the parametrisation $g = (v, \boldsymbol{x}) \in G \ltimes \mathfrak{g}^*$ the right-invariant one-form on $G \ltimes \mathfrak{g}^*$ is given by

$$\delta g g^{-1} = (\delta v v^{-1}, \delta\boldsymbol{x} - [\delta v v^{-1}, \delta\boldsymbol{x}]). \tag{A.1}$$

Similarly, the left-invariant one-form is

$$g^{-1}\delta g = (v^{-1}\delta v, \mathrm{Ad}^*(v)\delta\boldsymbol{x}). \tag{A.2}$$

We write an element of the Lie algebra of $G \ltimes \mathfrak{g}^*$ as $T = -(\boldsymbol{p}, \boldsymbol{k}) = -(p_a J^a, k_a P^a)$ and use the canonical inner product $\langle , \rangle$ on $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ to identify the dual $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$ with the Lie algebra $\mathfrak{g} \tilde{\oplus} \mathfrak{g}^*$. Then the canonical symplectic structure on the cotangent bundle of $G \ltimes \mathfrak{g}^*$ is

$$\Omega_0 = \frac{1}{2}(\langle \delta T, \delta g g^{-1}\rangle + \langle \delta\tilde{T}, g^{-1}\delta g\rangle), \tag{A.3}$$

where $\tilde{T} = g^{-1}Tg$. This expression is useful for understanding the relationship with the Heisenberg double. For calculations we use

$$\Omega_0 = \delta\langle T, \delta g g^{-1}\rangle = \delta\langle \tilde{T}, g^{-1}\delta g\rangle \tag{A.4}$$

to find

$$\Omega_0 = -\delta\left(\langle \boldsymbol{p}, \delta\boldsymbol{x}\rangle + \langle \boldsymbol{k} - [\boldsymbol{x}, \boldsymbol{p}], \delta v v^{-1}\rangle\right). \tag{A.5}$$

Two further symplectic structures can be obtained from the cotangent bundle by reduction. Pick a Cartan subalgebra of $G \ltimes \mathfrak{g}^*$, denote it by $\mathfrak{c}$ and and write $(\boldsymbol{\mu}, \boldsymbol{s})$ for an element in $\mathfrak{c}$. We obtain a symplectic structure on $G \ltimes \mathfrak{g}^* \times \mathfrak{c}$ from (A.3) by restricting $\tilde{T}$ to lie in $\mathfrak{c}$. Parametrising

$$\tilde{T} = -(\boldsymbol{\mu}, \boldsymbol{s}) \tag{A.6}$$

so that

$$T = -g(\boldsymbol{\mu}, \boldsymbol{s})g^{-1} \tag{A.7}$$

we now have the formulae

$$\boldsymbol{p} = \mathrm{Ad}(v)\boldsymbol{\mu} \tag{A.8}$$

$$\boldsymbol{k} = [\boldsymbol{x}, \boldsymbol{p}] + \mathrm{Ad}^*(v^{-1})\boldsymbol{s} \tag{A.9}$$

for $\boldsymbol{p}$ and $\boldsymbol{k}$ in terms of $g = (v, \boldsymbol{x})$ and $(\boldsymbol{\mu}, \boldsymbol{s})$. Inserting (A.6) and (A.7) into (A.3) one obtains

$$\omega_0 = -\delta\langle(\boldsymbol{\mu}, \boldsymbol{s}), g^{-1}\delta g\rangle = \delta\langle T, \delta g g^{-1}\rangle. \tag{A.10}$$

In terms of (A.8) and (A.9) this can be written as

$$\begin{aligned} \omega_0 &= -\delta(\langle \boldsymbol{s}, v^{-1}\delta v\rangle + \langle \boldsymbol{p}, \delta\boldsymbol{x}\rangle) \\ &= -\delta(\langle \boldsymbol{k}, \delta v v^{-1}\rangle - \langle \mathrm{Ad}(v)\delta\boldsymbol{\mu}, \boldsymbol{x}\rangle). \end{aligned} \tag{A.11}$$

If we require $(\boldsymbol{\mu}, \boldsymbol{s}) \in \mathfrak{c}$ to be constant, the form (A.11) becomes degenerate and ceases to be symplectic. It is, however, the pull-back of a symplectic form, namely the canonical symplectic form on the coadjoint orbit $\mathcal{O}_{\boldsymbol{\mu},\boldsymbol{s}} = \{g(\boldsymbol{\mu}, \boldsymbol{s})g^{-1} |\, g \in G \ltimes \mathfrak{g}^*\}$. These orbits are the symplectic leaves of the canonical Poisson structure on the dual of the Lie algebra of $G \ltimes \mathfrak{g}^*$.

A.2 Heisenberg double of $G \ltimes \mathfrak{g}^*$

The Heisenberg double of a Poisson-Lie group is a group with a Poisson structure, but it is not a Poisson-Lie group [14, 15]. The Heisenberg double $D_+(G \ltimes \mathfrak{g}^*)$ of the semi-direct product group $G \ltimes \mathfrak{g}^*$ can be identified with the Cartesian product $(G \ltimes \mathfrak{g}^*) \times (G \ltimes \mathfrak{g}^*)$ as a group. In order to describe its Poisson structure we need the dual Poisson-Lie group of $G \ltimes \mathfrak{g}^*$, which is isomorphic to the direct product $G \times \mathfrak{g}^*$ [7]. According to the general theory of Poisson-Lie groups [16] there exist group homomorphisms $S, S_\sigma : G \times \mathfrak{g}^* \to G \ltimes \mathfrak{g}^*$ such that

$$\begin{aligned} Z : G \times \mathfrak{g}^* &\to G \ltimes \mathfrak{g}^* \\ h^* &\mapsto S(h^*)(S_\sigma(h^*))^{-1} \end{aligned} \tag{A.12}$$

is a diffeomorphism, at least locally. In our case, we parametrise elements $h^* \in G \times \mathfrak{g}^*$ as $h^* = (u, -\boldsymbol{j})$ and find

$$S(u, -\boldsymbol{j}) = (u, 0) \quad S_\sigma(u, -\boldsymbol{j}) = (1, \boldsymbol{j}). \tag{A.13}$$

Hence

$$Z : (u, -\boldsymbol{j}) \mapsto S(u, -\boldsymbol{j})(S_\sigma(u, -\boldsymbol{j}))^{-1} = (u, -\mathrm{Ad}^*(u^{-1})\boldsymbol{j}), \tag{A.14}$$

which is a global diffeomorphism.

Now turning to the Heisenberg double, we have homomorphisms

$$\begin{array}{rcl} G \ltimes \mathfrak{g}^* & \rightarrow & D_+(G \ltimes \mathfrak{g}^*) \\ g & \mapsto & (g, g) \end{array} \tag{A.15}$$

and

$$\begin{array}{rcl} G \times \mathfrak{g}^* & \rightarrow & D_+(G \ltimes \mathfrak{g}^*) \\ h^* = (u, -\boldsymbol{j}) & \mapsto & (S(h^*), S_\sigma(h^*)) = ((u, 0), (1, \boldsymbol{j})). \end{array} \tag{A.16}$$

The Heisenberg double of $G \ltimes \mathfrak{g}^*$ is factorisable, so we can define further elements $h = (w, \boldsymbol{y}) \in G \ltimes \mathfrak{g}^*$ and $g^* = (\tilde{u}, -\tilde{\mathbf{j}}) \in G \times \mathfrak{g}^*$ via the relation

$$gg^* = h^* h \tag{A.17}$$

in $D_+(G \ltimes \mathfrak{g}^*)$. In particular this implies that h^* is related to g^* via a dressing transformation, which is the $G \ltimes \mathfrak{g}^*$-conjugation action on the $G \ltimes \mathfrak{g}^*$-element associated via Z:

$$g(Z(g^*))g^{-1} = Z(h^*) \tag{A.18}$$

or, in components:

$$u = v\tilde{u}v^{-1} \quad \boldsymbol{j} = (1 - \mathrm{Ad}^*(u))\boldsymbol{x} + \mathrm{Ad}^*(v^{-1})\tilde{\mathbf{j}}. \tag{A.19}$$

For the $G \ltimes \mathfrak{g}^*$ elements h and g, the relation (A.17) implies

$$w = v \qquad \boldsymbol{y} = \mathrm{Ad}^*(u)\boldsymbol{x}. \tag{A.20}$$

Following the notation used by Alekseev and Malkin in [15] , the symplectic structure on the Heisenberg double is

$$\Omega_H \;=\; \frac{1}{2}\left(\langle \delta h^*(h^*)^{-1}, \delta g g^{-1}\rangle + \langle (g^*)^{-1}\delta g^*, h^{-1}\delta h\rangle\right), \tag{A.21}$$

and with our parametrisation of the elements h, h^*, g, g^* we have

$$\begin{aligned} \Omega_H &= \frac{1}{2}(\langle(\delta u u^{-1}, -\delta\boldsymbol{j}), \delta g g^{-1}\rangle + \langle(\tilde{u}^{-1}\delta\tilde{u}, -\delta\tilde{\mathbf{j}}), h^{-1}\delta h\rangle) \\ &= \frac{1}{2}(\langle \delta u u^{-1}, \delta\boldsymbol{x} - [\delta v v^{-1}, \boldsymbol{x}]\rangle - \langle \delta\boldsymbol{j}, \delta v v^{-1}\rangle \\ &\quad + \langle \tilde{u}^{-1}\delta\tilde{u}, \mathrm{Ad}^*(w)\delta\boldsymbol{y}\rangle - \langle \delta\tilde{\mathbf{j}}, w^{-1}\delta w\rangle). \end{aligned} \tag{A.22}$$

Using (A.19) and (A.20) to write g^* and h in terms of g and h^* we find, after some algebra:

$$
\begin{aligned}
\Omega_H &= -\delta\left(\langle \boldsymbol{j} - (1-\mathrm{Ad}^*(u))\boldsymbol{x}, \delta v v^{-1}\rangle + \langle \boldsymbol{x}, \delta u u^{-1}\rangle\right) \\
&= -\delta\left(\langle \boldsymbol{j}, \delta v v^{-1}\rangle + \langle \mathrm{Ad}^*(v)\boldsymbol{x}, \delta\tilde{u}\tilde{u}^{-1}\rangle\right).
\end{aligned}
\tag{A.23}
$$

Note that we recover the symplectic structure (A.5) of the cotangent bundle $T^*(G \ltimes \mathfrak{g}^*)$ if we expand $u = e^{-p_a J^a}$ and keep linear terms in $\boldsymbol{p}$.

In analogy with the cotangent bundle symplectic structure we briefly describe two symplectic structures derived from the Heisenberg double structure. The first is defined on the product of $G \ltimes \mathfrak{g}^*$ with the Cartan subalgebra $\mathfrak{c}$, assumed to be abelian. Restricting h^* to be of the form $(\tilde{u}, -\tilde{\mathbf{j}}) = (e^{-\boldsymbol{\mu}}, -\boldsymbol{s})$ with $(\boldsymbol{\mu}, \boldsymbol{s}) \in \mathfrak{c}$ arbitrary yields

$$
\begin{aligned}
\omega_H &= -\delta(\langle \boldsymbol{s}, v^{-1}\delta v\rangle + \langle \boldsymbol{x}, \delta u u^{-1}\rangle) \\
&= -\delta(\langle \boldsymbol{j}, \delta v v^{-1}\rangle - \langle \boldsymbol{x}, \mathrm{Ad}(v)\delta\boldsymbol{\mu}\rangle),
\end{aligned}
\tag{A.24}
$$

where now

$$
\boldsymbol{j} = (1-\mathrm{Ad}^*(u))\boldsymbol{x} + \mathrm{Ad}^*(v^{-1})\boldsymbol{s} \quad \text{and} \quad u = v e^{-\boldsymbol{\mu}} v^{-1}. \tag{A.25}
$$

Finally, we obtain a degenerate two-form by setting $\boldsymbol{\mu}$ and $\boldsymbol{s}$ to constant values. This two-form is the pull-back of a canonical symplectic structure on the conjugacy classes of $G \ltimes \mathfrak{g}^*$ [7]. According to the general theory of Poisson-Lie groups [16] these conjugacy classes are the symplectic leaves of the dual Poisson-Lie group $G \times \mathfrak{g}^*$. They are analogous to the coadjoint orbits discussed at the end of the previous subsection.

B Alekseev and Malkin's description of the symplectic structure

In [8], Alekseev and Malkin study the moduli space of flat H-connections for simple complex Lie groups H (or their compact real forms) on a two-dimensional surface $S_{g,n}$ of genus g with n punctures. Most of their results apply also to the case $H = G \ltimes \mathfrak{g}^*$ studied in the present paper. In this appendix we summarise some of the results from [8] required in our calculations, using our notation, and indicate modifications where necessary. The first applies without change to our situation.

Lemma B.1 *(Lemma 2 [8])*

Let $P \subset S_{g,n}$ be region of $S_{g,n}$ on which the H-connection A_S takes the form

$$
A_S = \gamma B \gamma^{-1} + \gamma d\gamma^{-1} \tag{B.1}
$$

where B is a gauge field on P with curvature $F_B = dB + B \wedge B$ and $\gamma : S_{g,n} \to H$. Then, the canonical symplectic form on P is given by

$$
\int_P \langle \delta A_S \wedge \delta A_S\rangle = \int_P \langle \delta B \wedge \delta B\rangle + 2\delta\langle F_B\,,\, \delta\gamma^{-1}\gamma\rangle + \int_{\partial P} \langle \delta\gamma^{-1}\gamma\,,\, d(\delta\gamma^{-1}\gamma)\rangle - 2\delta\langle B\,,\, \delta\gamma^{-1}\gamma\rangle. \tag{B.2}
$$

The other results we require from [8] concern the evaluation of the contribution of the ordinary punctures and handles to the two-form (4.14). As explained in Sect. 4, these contributions can be expressed as boundary integrals along the curves m_i, a_j, b_j, which can be evaluated by means of a continuity condition and related to the corresponding holonomies M_i, A_j, B_j. We summarise the results obtained in [8] and sketch their derivation.

We start by considering the ordinary punctures. Expression (4.14) for the two-form Ω involves two integrals along the curve m_i. The sum of these, in the following denoted by Ω_{M_i} is the contribution of the i-the puncture to the symplectic structure. Again the result applies to $H = G \ltimes \mathfrak{g}^*$ without change and is given by the following lemma

Lemma B.2 *(Lemma 3, [8])*

Let

$$\Omega_{M_i} = \int_{m_i} \langle \delta\gamma_0^{-1}\gamma_0 d\left(\delta\gamma_0^{-1}\gamma_0\right)\rangle - \int_{m_i} \langle \delta\gamma_{M_i}^{-1}\gamma_{M_i} d\left(\delta\gamma_{M_i}^{-1}\gamma_{M_i}\right)\rangle - \tfrac{2}{k}\delta\langle D_i \delta\gamma_{M_i}^{-1}\gamma_{M_i}\rangle d\phi_i \tag{B.3}$$

with γ_0, γ_{M_i} related by the overlap condition

$$\gamma_0^{-1}|_{m_i} = N_{M_i} D_{M_i}(\phi_i)\gamma_{M_i}^{-1}|_{m_i}, \tag{B.4}$$

where $C_{M_i}(\phi_i) = \exp(\frac{1}{k}D_i\phi_i)$ and N_{M_i} is an arbitrary but constant element of the gauge group $G \ltimes \mathfrak{g}^*$. Denote starting and endpoint of m_i by p_{i-1} and p_i and define

$$K_i = \gamma_0^{-1}(p_i) \quad K_{i-1} = \gamma_0^{-1}(p_{i-1}) \qquad\qquad g_i = \gamma_{M_i}(p_i) = \gamma_{M_i}(p_{i-1}). \tag{B.5}$$

Then, the holonomy M_i along m_i is given by

$$M_i = g_i C_i g_i^{-1} = K_i^{-1}K_{i-1}, \tag{B.6}$$

and the two-form (B.3) *can be expressed as*

$$\Omega_{M_i} = \frac{k}{4\pi}\langle C_i g_i^{-1}\delta g_i C_i^{-1} \wedge g_i^{-1}\delta g_i\rangle - \langle \delta K_i K_i^{-1} \wedge \delta K_{i-1}K_{i-1}^{-1}\rangle. \tag{B.7}$$

Proof [8]:

The overlap condition (B.4) implies

$$\begin{aligned}\langle \delta\gamma_0^{-1}\gamma_0 d\left(\delta\gamma_0^{-1}\gamma_0\right)\rangle|_{m_i} =& d\langle \delta N_{M_i}N_{M_i}^{-1}\ \delta\gamma_0^{-1}\gamma_0\rangle|_{m_i} + \langle \delta\gamma_{M_i}^{-1}\gamma_{M_i} d\left(\delta\gamma_{M_i}^{-1}\gamma_{M_i}\right)\rangle|_{m_i} \\ &- \tfrac{2}{k}\delta\langle D_i d\phi_i\ \delta\gamma_{M_i}^{-1}\gamma_{M_i}\rangle|_{m_i}.\end{aligned} \tag{B.8}$$

Using this identity in (B.3), one obtains

$$\Omega_{M_i} = \left[\langle \delta N_{M_i}N_{M_i}^{-1} \wedge \delta\gamma_0^{-1}\gamma_0\rangle\right]_{p_{i-1}}^{p_i} = -\left[\langle \gamma_{M_i}\delta\gamma_{M_i}^{-1} \wedge \gamma_0\delta\gamma_0^{-1}\rangle\right]_{p_{i-1}}^{p_i}, \tag{B.9}$$

where the last equality follows by using the overlap condition (B.8) to evaluate the term $\delta N_{M_i}N_{M_i}^{-1}$. Denoting the values of γ_0 and γ_{M_i} at the endpoints by, respectively, K_i, K_{i-1} and g_i as in (B.5) and applying (B.6) repeatedly one obtains expression (B.7). $\square$

Next we turn to the contribution of the handles. For each handle, there are two non-contractible loops a_j and b_j whose homotopy classes are generators of the fundamental group $\pi_1(S^\infty_{g,n}, p_0)$. When cutting $S^\infty_{g,n}$ to produce the Polygon P_0 each of these curves appears twice in the boundary of P_0, with opposite orientation. As in the main text after (4.18) we denote a generic generator $\in \{a_1, b_1, \ldots, a_g, b_g\}$ by x and use the notation γ_0' and γ_0'' for the restriction of trivialising gauge transformation γ_0 to the two boundary components of P_0 corresponding to x. γ_0' and γ_0'' then satisfy the continuity condition

$$(\gamma_0')^{-1}|_x = N_x\,(\gamma_0'')^{-1}|_x \quad \text{with} \quad dN_x = 0, \tag{B.10}$$

and, denoting the endpoints of x by y_1, y_2, the contribution to the two-form (4.14) is given by

$$\Omega_X = \int_{y_1}^{y_2} \langle \delta\,(\gamma_0')^{-1}\,\gamma_0' d\left(\delta\,(\gamma_0')^{-1}\,\gamma_0'\right)\rangle - \int_{y_1}^{y_2} \langle \delta\,(\gamma_0'')^{-1}\,\gamma_0'' d\left(\delta\,(\gamma_0'')^{-1}\,\gamma_0''\right)\rangle. \tag{B.11}$$

The full contribution Ω_{H_i} of a handle is obtained by adding (B.11) for the corresponding a- and b-cycle and can be evaluated as follows.

Lemma B.3 *(Lemma 4, [8])*
Let Ω_{A_i} and Ω_{B_i} be defined according to (B.11) *with endpoints as given in Fig. 3 and set*

$$\Omega_{H_i} = \Omega_{A_i} + \Omega_{B_i}. \tag{B.12}$$

Let the values of the function γ_0 at the endpoints p_i be given as in (4.23)

$$\begin{aligned}
\gamma_0(p_{n+4j-3}) &= K^{-1}_{n+2j-1} := A_j[B_{j-1}, A^{-1}_{j-1}]\cdots[B_1, A_1^{-1}]M_n\cdots M_1\\
\gamma_0(p_{n+4j-2}) &= B_j^{-1}A_j[B_{j-1}, A^{-1}_{j-1}]\cdots[B_1, A_1^{-1}]M_n\cdots M_1\\
\gamma_0(p_{n+4j-1}) &= A_j^{-1}B_j^{-1}A_j[B_{j-1}, A^{-1}_{j-1}]\cdots[B_1, A_1^{-1}]M_n\cdots M_1\\
\gamma_0(p_{n+4j}) &= K^{-1}_{n+2j} := [B_j, A_j^{-1}]\cdots[B_1, A_1^{-1}]M_n\cdots M_1 \quad \textit{for } 1\leq j\leq g,
\end{aligned} \tag{B.13}$$

and parametrise the holonomies A_i, B_i, $i = 1, \ldots, g$ in terms of variables $g_{n+2i-1}, g_{n+2i} \in H$ and C_{n+i} in a fixed Cartan subgroup via

$$\begin{aligned}
A_i &= \gamma_0(p_{n+4i-3})\gamma_0^{-1}(p_{n+4(i-1)}) = \gamma_0(p_{n+4i-2})\gamma_0^{-1}(p_{n+4i-1}) = g_{n+2i-1}C^{-1}_{n+i}g^{-1}_{n+2i-1}\\
B_i &= \gamma_0(p_{n+4i})\gamma_0^{-1}(p_{n+4i-1}) = \gamma_0(p_{n+4i-3})\gamma_0^{-1}(p_{n+4i-2}) = g_{n+2i}g^{-1}_{n+2i-1}.
\end{aligned} \tag{B.14}$$

Then, the two-form (B.12) *is given by*

$$\begin{aligned}
\Omega_{H_i} =&\Omega_{A_i} + \Omega_{B_i}\\
=&\frac{k}{4\pi}\langle C_{n+i}g^{-1}_{n+2i-1}\delta g_{n+2i-1}C^{-1}_{n+i}\wedge g^{-1}_{n+2i-1}\delta g_{n+2i-1}\rangle\\
+&\frac{k}{4\pi}\langle C^{-1}_{n+i}g^{-1}_{n+2i}\delta g_{n+2i}C_{n+i}\wedge g^{-1}_{n+2i}\delta g_{n+2i}\rangle\\
-&\frac{k}{2\pi}\langle \delta C_{n+i}C^{-1}_{n+i}\wedge\left(g^{-1}_{n+2i-1}\delta g_{n+2i-1} - g^{-1}_{n+2i}\delta g_{n+2i}\right)\rangle\\
-&\frac{k}{4\pi}\left(\langle\delta K_{n+2i-1}K^{-1}_{n+2i-1}\wedge\delta K_{n+2(i-1)}K^{-1}_{n+2(i-1)}\rangle + \langle\delta K_{n+2i}K^{-1}_{n+2i}\wedge\delta K_{n+2i-1}K^{-1}_{n+2i-1}\rangle\right).
\end{aligned} \tag{B.15}$$

Proof [8]:

The overlap condition (B.10) implies

$$\langle \delta\,(\gamma_0')^{-1}\,\gamma_0' d(\delta\,(\gamma_0')^{-1}\,\gamma_0')\rangle|_x = d\langle \delta N_x N_x^{-1}, \delta\,(\gamma_0')^{-1}\,\gamma_0'\rangle|_x + \langle \delta\,(\gamma_0'')^{-1}\,\gamma_0'', d(\delta\,(\gamma_0'')^{-1}\,\gamma_0'')\rangle|_x \tag{B.16}$$

and inserting this expression into (B.11) gives

$$\Omega_X = \left[\langle \delta N_x N_x^{-1} \wedge \delta\,(\gamma_0')^{-1}\,\gamma_0'\rangle\right]_{y_1}^{y_2} = -\left[\langle \gamma_0''\delta\,(\gamma_0'')^{-1} \wedge \gamma_0'\delta\,(\gamma_0')^{-1}\rangle\right]_{y_1}^{y_2}. \tag{B.17}$$

After a rather lengthy calculation which repeatedly makes use of (B.14), (4.23) and adding the contributions of the generators a_i and b_i one obtains expression (B.18). □

As explained in Sect. 2, in groups of the form $G \ltimes \mathfrak{g}^*$ we cannot assume without loss of generality that the holomomy variables A_i and B_i, $i = 1, \ldots, g$ can be parametrised in terms of elements of a given Cartan subgroup, or even elements of a finite union of Cartan subgroups, via (B.14). We have therefore repeated the calculations without making use of such a parametrisation. The result is

Lemma B.4 *With notation in Lemma B.3, the the two-form* (B.12) *can be written as*

$$\begin{aligned}\Omega_{H_i} =& \Omega_{A_i} + \Omega_{B_i} \\ =& -\frac{k}{4\pi}\sum_{i=1}^{g}\left(\langle A_i^{-1}\delta A_i \wedge B_i^{-1}\delta B_i\rangle + \langle \delta(B_i A_i B_i^{-1}) B_i A_i^{-1} B_i^{-1} \wedge \delta B_i\, B_i^{-1}\rangle\right) \\ & -\frac{k}{4\pi}\left(\langle \delta K_{n+2i-1}K_{n+2i-1}^{-1} \wedge \delta K_{n+2i-2}K_{n+2i-2}^{-1}\rangle + \langle \delta K_{n+2i}K_{n+2i}^{-1} \wedge \delta K_{n+2i-1}K_{n+2i-1}^{-1}\rangle\right).\end{aligned} \tag{B.18}$$

This is the formula used in the main text of the paper.

References

[1] E. Witten, Quantum Field Theory and the Jones Polynomial, Commun. Math. Phys. 121 (1989) 351–399.

[2] E. Guadagnini, The Link invariants of the Chern-Simons field theory, de Gruyter, Berlin, 1993.

[3] M. Atiyah, The geometry and physics of knots, Cambridge University Press, Cambridge, 1990.

[4] A. Achucarro, P. Townsend, A Chern–Simons action for three-dimensional anti-de Sitter supergravity theories, Phys. Lett. B 180 (1986) 85–100.

[5] E. Witten, 2+1 dimensional gravity as an exactly soluble system, Nucl. Phys. B 311 (1988) 46–78.

[6] C. Meusburger and B. J. Schroers, Boundary conditions and symplectic structure in the Chern-Simons formulation of (2+1)-dimensional gravity, EMPG-05-08

[7] C. Meusburger and B. J. Schroers, The quantisation of Poisson structures arising in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$, Adv. Theor. Math. Phys. 7 (2004) 1003–1043.

[8] A. Y. Alekseev and A. Z. Malkin, Symplectic structure of the moduli space of flat connections on a Riemann surface, Commun. Math. Phys. 169 (1995) 99–119.

[9] J. E. Marsden, T. S. Ratiu, Introduction to mechanics and symmetry, Springer Verlag, New York, 1999.

[10] V. S. Varadarajan, Lie groups, Lie algebras and their representation, Springer Verlag, New York, 1984.

[11] A. W. Knapp, Lie groups Beyond an introduction, Birkhäuser, Boston, 1996.

[12] A. L. Onishchik and E. B. Vinberg (Eds), Lie groups and Lie algebras III, Ecyclopaedia of Mathematical Sciences Vol 41, Springer Verlag, Berlin Heidelberg, 1994

[13] F. Falceto and K. Gawedzki, Boundary G/G theory and topological Poisson-Lie sigma model, Lett. Math. Phys. 59 (2002) 61–79.

[14] M. A. Semenov-Tian-Shansky, Dressing transformations and Poisson-Lie group actions, Publ. Res. Inst. Math. Sci. Kyoto University 21 (1996) 1237–60.

[15] A. Y. Alekseev, A. Z. Malkin, Symplectic structures associated to Lie-Poisson groups, Commun. Math. Phys. 162 (1994) 147-173.

[16] V. Chari, A. Pressley, Quantum Groups, Cambridge University Press, Cambridge, 1994.

[17] R. Sjamaar and E. Lerman, Stratified symplectic spaces and reduction, Ann. Math. 134 (1991) 375–422.

[18] K. Ezawa, Chern-Simons quantization of (2+1) anti-de Sitter gravity on a torus, Class. Quant. Grav. 12 (1995) 373–392.